# Approach to equilibrium in translation-invariant quantum spin systems

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PhD Minicourse
Politecnico di Milano
November 2025

# Outline

Part 1: Main Results

- Refresher on quantum spin systems
- Approach to thermal equilibrium
- Constrained approach to equilibrium

Part 2: Main Tools

- Structural theory of constants of motion
- Dynamical conservation laws
- Partial solutions to Conjectures

Part 3: Approach to Equilibrium in non-integrable systems

- Japanese School of non-integrable systems
- Link between the two research programs
- Results for the Ising chain and other models

# Quantum lattice spin systems

# Quantum lattice spin systems

- Lattice $\mathbb{Z}^d$, family of translations $\{\tau_x\}_{x\in\mathbb{Z}^d}$, family $\mathcal{F}$ of **finite subsets** of $\mathbb{Z}^d$
- Fixed Hilbert space $\mathcal{H}_0 := \mathbb{C}^N$

- $\mathcal{H}_x := \mathcal{H}_0$ for every $x \in \mathbb{Z}^d$
- $\mathcal{H}_X := \otimes_{x\in X}\mathcal{H}_x$ for $X \in \mathcal{F}$, and $\mathcal{U}_X :=$ bounded operators on $\mathcal{H}_X$
- $\mathcal{U}_X \ni A \mapsto A \otimes 1_{\tilde{X}\setminus X} \in \mathcal{U}_{\tilde{X}}$ for $X \subset \tilde{X} \in \mathcal{F}$

- **Local observables** $\mathcal{U}_{\mathrm{loc}} := \bigcup_{X\in\mathcal{F}}\mathcal{U}_X$
- Spin C*-algebra $\mathcal{U}$ is the norm-completion of $\mathcal{U}_{\mathrm{loc}}$
- **Translation invariant states:** $\mathcal{S}_{\mathrm{I}} = \{\rho\colon \mathcal{U} \to \mathbb{C} \mid \rho$ positive, linear, $\rho(1) = 1$ and $\rho\circ\tau_x = \rho \;\; \forall x \in \mathbb{Z}^d\}$

# Interactions, local Hamiltonians, dynamics

- **Interaction**: family $\{\Phi(X)\}_{X\in\mathcal{F}}$ such that $\Phi(X)\in\mathcal{U}_X$ is self-adjoint.

  We always assume translation invariance: $\tau_x(\Phi(X))=\Phi(X+x)\ \ \forall x\in\mathbb{Z}^d\ \forall X\in\mathcal{F}$

- $H_\Phi(\Lambda)=\sum_{X\subset\Lambda}\Phi(X)$ is the **local Hamiltonian** on $\Lambda\in\mathcal{F}$

- **Local dynamics** on $\Lambda$:
$$\alpha^t_{\Phi,\Lambda}(A)=\mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)}A\mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)},\quad A\in\mathcal{U}_\Lambda$$

- We say the **(global) dynamics exists** if for all $A\in\mathcal{U}$ the limit
$$\alpha^t_\Phi(A):=\lim_{\Lambda\uparrow\mathbb{Z}^d}\alpha^t_{\Phi,\Lambda}(A)$$
  exists and is uniform for $t$ in compact sets, where $\Lambda\uparrow\mathbb{Z}^d$ denotes the limit over an increasing and exhaustive family of cubes in $\mathbb{Z}^d$ centred at 0

- **Time evolution of a state**: $\rho_t:=\rho\circ\alpha^t_\Phi$

## Spaces of interactions

- **Big space** of interactions $\mathcal{B}_{\mathrm{b}} = \{\Phi : \|\Phi\|_{\mathrm{b}} < \infty\}$ where

$$\|\Phi\|_{\mathrm{b}} = \sum_{X \ni \mathbf{0}} \frac{\|\Phi(X)\|}{|X|}$$

- **Small space** of interactions $\mathcal{B}_{\mathrm{s}} = \{\Phi : \|\Phi\|_{\mathrm{s}} < \infty\}$ where

$$\|\Phi\|_{\mathrm{s}} = \sum_{X \ni \mathbf{0}} \|\Phi(X)\|$$

- **Finite range** interactions:

$$\mathcal{B}_{\mathrm{f}} = \{\Phi : \exists R \in \mathbb{N} \ \mathrm{diam}(X) > R \ \Rightarrow \ \Phi(X) = 0\}$$

We have $\mathcal{B}_{\mathrm{f}} \subset \mathcal{B}_{\mathrm{s}} \subset \mathcal{B}_{\mathrm{b}}$, both $\mathcal{B}_{\mathrm{s}}$ and $\mathcal{B}_{\mathrm{b}}$ are Banach spaces, and $\mathcal{B}_{\mathrm{f}}$ is a dense subset of each

- $\alpha_\Phi$ may not exist for $\Phi \in \mathcal{B}_{\mathrm{s}}$
- $\alpha_\Phi$ does exist for $\Phi \in \mathcal{B}_f$

# Gibbs variational principle and Equilibrium States

Let $\Phi \in \mathcal{B}_{\mathrm{b}}$ and $\rho \in \mathcal{S}_{\mathrm{I}}$. Notation: $\rho_\Lambda \in \mathcal{U}_\Lambda$ satisfies $\rho(A) = \mathrm{tr}(\rho_\Lambda A)$ for all $A \in \mathcal{U}_\Lambda$ (density matrix)

- **Specific entropy** of $\rho$:

$$s(\rho) := -\lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \mathrm{tr}(\rho_\Lambda \log \rho_\Lambda) \in [0, \log N]$$

It is affine & upper semi-continuous

- **Specific energy** of $\Phi$:

$$E_\Phi := \sum_{X \ni 0} \frac{\Phi(X)}{|X|} \in \mathcal{U}$$

It satisfies $\lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \rho(H_\Phi(\Lambda)) = \rho(E_\Phi)$ for all $\rho \in \mathcal{S}_{\mathrm{I}}$.

- **Pressure** of $\Phi$:

$$P(\Phi) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \log(\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})) < \infty$$

**Gibbs variational principle:** $P(\beta\Phi) = \sup_{\rho \in \mathcal{S}_{\mathrm{I}}} \big(s(\rho) - \beta\rho(E_\Phi)\big)$ for inverse temperature $\beta$

Maximizers are the **equilibrium states**: $\mathcal{S}_{\mathrm{eq}}(\beta\Phi) = \{\rho \in \mathcal{S}_{\mathrm{I}} \mid P(\beta\Phi) = s(\rho) - \beta\rho(E_\Phi)\}$

**Dual variational principle:** $s(\rho) = \inf_{\Phi \in \mathcal{B}_{\mathrm{b}}} \big(P(\beta\Phi) + \beta\rho(E_\Phi)\big)$

# Surface energies

**Surface energies** for $\Lambda \in \mathcal{F}$ of an interaction $\Phi$ are defined as

$$W_\Phi(\Lambda) = \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi(X)$$

$$= \lim_{\Lambda' \uparrow \mathbb{Z}^d} \left(H_\Phi(\Lambda') - H_\Phi(\Lambda) - H_\Phi(\Lambda' \setminus \Lambda)\right)$$

On physical grounds, surface energies should play a central role in the study of approach to equilibrium

- Surface energies may not exist for $\Phi \in \mathcal{B}_\mathrm{b}$
- Surface energies do exist for $\Phi \in \mathcal{B}_\mathrm{s}$

**Proposition**

Let $\Phi \in \mathcal{B}_\mathrm{s}$. Then $W_\Phi(\Lambda)$ exists for every $\Lambda \in \mathcal{F}$, and $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} W_\Phi(\Lambda) = 0$.

# $\mathcal{B}_{\mathrm{sd}}$ space of *physical* interactions

For $\Phi \in \mathcal{B}_{\mathrm{s}}$ consider the $*$-derivation $\delta_\Phi : \mathcal{U}_{\mathrm{loc}} \to \mathcal{U}$ defined by

$$\delta_\Phi(A) = \sum_{X \in \mathcal{F}} [\Phi(X), A] = \lim_{\Lambda \uparrow \mathbb{Z}^d} [H_\Phi(\Lambda), A], \qquad A \in \mathcal{U}_{\mathrm{loc}}$$

It is closable and we denote its closure again by $\delta_\Phi$.

**Definition of $\mathcal{B}_{\mathrm{sd}}$**

$$\mathcal{B}_{\mathrm{sd}} = \{\Phi \in \mathcal{B}_{\mathrm{s}} \ : \ \delta_\Phi \text{ generates dynamics } \alpha_\Phi \text{ on } \mathcal{U}\}$$

**Theorem**

$\delta_\Phi$ generates dynamics $\alpha_\Phi$ on $\mathcal{U}$ if and only if $(\mathrm{i} \pm \delta_\Phi)\mathcal{U}_{\mathrm{loc}}$ is dense in $\mathcal{U}$. In that case

$$\alpha_\Phi^t(A) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)} A \mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)}, \qquad A \in \mathcal{U},$$

where the limit is uniform for $t$ in compacts.

# $\mathcal{B}_{\mathrm{sd}}$ space of *physical* interactions

The naturalness of $\mathcal{B}_{\mathrm{sd}}$ stems also from the following result

**Theorem**

Suppose that $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$. Then the following statements are equivalent:

- $\Phi$ and $\Psi$ are equivalent – *to be understood intuitively for now*
- $\mathcal{S}_{\mathrm{eq}}(\Phi) \cap \mathcal{S}_{\mathrm{eq}}(\Psi) \neq \emptyset$
- $\mathcal{S}_{\mathrm{eq}}(\Phi) = \mathcal{S}_{\mathrm{eq}}(\Psi)$
- $\alpha_\Phi = \alpha_\Psi$

In contrast, $\mathcal{B}_{\mathrm{b}}$ contains non-equivalent interactions that share an equilibrium state, which is pathological

Important classes of interactions are in $\mathcal{B}_{\mathrm{sd}}$ which we will see later on:

- The well-known space of **exponentially-decaying interactions**: $\mathcal{B}^r \subset \mathcal{B}_{\mathrm{sd}}$ for $r > 0$.
- We will also work with the space of **diameter-interactions**: $\mathcal{B}_\gamma^{\mathrm{diam}} \subset \mathcal{B}_{\mathrm{sd}}$ for $\gamma > d$.

# Regularity

**Specific relative entropy** of $\rho \in \mathcal{S}_\mathrm{I}$ with respect to $\omega \in \mathcal{S}_\mathrm{I}$ (assuming the limit exists):

$$s(\rho|\omega) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\rho_\Lambda(\log \rho_\Lambda - \log \omega_\Lambda)) \geq 0$$

**Definition**

A pair $(\omega, \Psi) \in \mathcal{S}_\mathrm{I} \times \mathcal{B}_\mathrm{sd}$ is called **regular** if the relative entropy $s(\rho|\omega)$ exists for all $\rho \in \mathcal{S}_\mathrm{I}$ and satisfies the **entropy balance equation**:

$$s(\rho|\omega) = -s(\rho) + \rho(E_\Psi) + P(\Psi).$$

- $(\omega, \Psi)$ is regular $\Longrightarrow \omega \in \mathcal{S}_\mathrm{eq}(\Psi)$
- **R-Conjecture:** $(\omega, \Psi)$ is regular for every $\Psi \in \mathcal{B}_\mathrm{sd}$ and $\omega \in \mathcal{S}_\mathrm{eq}(\Psi)$
- It is a very hard open problem involving the structural aspects of QSS.
- **Thm:** If $\Psi \in \mathcal{B}^r$ with $\|\Psi\|_r < r$, or $\Psi \in \mathcal{B}_\mathrm{f}$ in $d = 1$, then $(\omega, \Psi)$ is regular for every $\omega \in \mathcal{S}_\mathrm{eq}(\Psi)$

[Jakšić, Pillet, Tauber'24]

# Approach to Equilibrium

$$\omega \longrightarrow \overline{\omega}_T \longrightarrow \omega_+ \stackrel{?}{=} \nu_{\mathrm{eq}}$$

# Equilibrium Steady States (ESS)

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. For $T > 0$ define

$$\overline{\omega}_T = \frac{1}{T}\int_0^T \omega \circ \alpha_\Phi^t \, \mathrm{d}t$$

and consider the set of **Equilibrium Steady States** (ESS):

$$\mathcal{S}_+(\omega, \Phi) = \{\text{weak*-limit points of } (\overline{\omega}_T)_{T>0} \text{ as } T \to \infty\}.$$

- $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ iff $\omega_+ = \lim_{n\to\infty} \overline{\omega}_{T_n}$ for some subseqence $T_n \uparrow \infty$
- $\mathcal{S}_+(\omega, \Phi) = \{\omega_+\}$ iff $\omega_+ = \lim_{T\to\infty} \overline{\omega}_T$
- $\omega_+$ is $\alpha_\Phi$-invariant
- $\omega_+ = \omega$ iff $\omega$ is $\alpha_\Phi$-invariant, which is a trivial setting (which we always exclude)

# Basic Conservation Laws

**We postulate the conservation of specific entropy and energy:**

$$\forall \omega \in \mathcal{S}_{\mathrm{I}}\ \forall t > 0 \qquad s(\omega \circ \alpha_\Phi^t) = s(\omega) \qquad \text{and} \qquad (\omega \circ \alpha_\Phi^t)(E_\Phi) = \omega(E_\Phi)$$

Fix the initial state $\omega \in \mathcal{S}_{\mathrm{I}}$. For the time average we get:

$$\forall T > 0 \qquad s(\omega) = s(\overline{\omega}_T) \qquad \text{and} \qquad \omega(E_\Phi) = \overline{\omega}_T(E_\Phi)$$

So for any $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$:

$$s(\omega) \leq s(\omega_+) \qquad \text{and} \qquad \omega(E_\Phi) = \omega_+(E_\Phi).$$

- $E_\Phi$ is a **constant of motion** of the dynamics $\alpha_\Phi$
- *Ruelle problem*: when does $s(\omega) < s(\omega_+)$ hold?
- We will further discuss these (and other) conservation laws tomorrow.

# Approach to Thermal Equilibrium – definition

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Consider the initial specific energy $e_0 := \omega(E_\Phi)$. We know it is preserved.

**Setting the equilibrium inverse temperature $\beta_*$**

We set such $\beta_*$ that for some $\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*\Phi)$ we have $\nu_{\mathrm{eq}}(E_\Phi) = e_0$, provided such $\beta_*$ exists.

If such $\beta_*$ exists, it is unique. If it does not exist, approach to thermal equilibrium is impossible.

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If such $\beta_*$ exists, it is unique. If it does not exist, approach to thermal equilibrium is impossible.

**Definition of Approach to Thermal Equilibrium**

The pair $(\omega, \Phi)$ has the property of Approach to Thermal Equilibrium if

(1) $\mathcal{S}_+(\omega, \Phi) = \{\omega_+\}$

(2) $\omega_+ \in \mathcal{S}_\mathrm{eq}(\beta_*\Phi)$

- The dynamical problem (1) can be answered only in the context of specific models.
- We focus on the structural theory of (2), developing it for $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$.

# Constants of Motion & Admissible States

Consider the real vector space $\mathcal{U}_{\mathrm{real}}$ of self-adjoint elements of $\mathcal{U}$.

**Constant of Motion**

We call $C \in \mathcal{U}_{\mathrm{real}}$ a **constant of motion** for $\Phi \in \mathcal{B}_{\mathrm{sd}}$ when

$$\rho \circ \alpha_\Phi^t(C) = \rho(C) \qquad \forall \rho \in \mathcal{S}_{\mathrm{I}} \ \forall t \in \mathbb{R}$$

We denote by $\mathfrak{C} = \mathfrak{C}(\alpha_\Phi)$ the set of all constants of motion for $\Phi$.

- There is a natural equivalence structure in $\mathfrak{C}$
- Recall $E_\Phi \in \mathfrak{C}$. The choice of $\beta_*$ guarantees $\omega(E_\Phi) = e_0 = \nu_{\mathrm{eq}}(E_\Phi)$
- Suppose that for each $\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*\Phi)$ satisfying $e_0 = \nu_{\mathrm{eq}}(E_\Phi)$ there exists $C \in \mathfrak{C}$ such that $\omega(C) \neq \nu_{\mathrm{eq}}(C)$ Approach to Thermal Equilibrium is then impossible!

**Admissible States**

The initial state $\omega$ is called **admissible** for $\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*\Phi)$ when $\omega(C) = \nu_{\mathrm{eq}}(C)$ for all $C \in \mathfrak{C}$.

## Approach to Thermal Equilibrium – assumptions

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Recall we assume the basic conservation laws, in particular $E_\Phi \in \mathfrak{C}$.

**Theorem** — Jakšić, Pillet, S, Tauber '25

Assume that $\mathcal{S}_{\mathrm{eq}}(\beta_*\Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_*\Phi)$ regular, and that $\omega$ is admissible for $\nu_{\mathrm{eq}}$.
For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

(1) $\omega_+ = \nu_{\mathrm{eq}}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that $(\omega_+, \Psi_+)$ is regular, and $E_{\Psi_+} \in \mathfrak{C}$

- Regularity of $(\nu_{\mathrm{eq}}, \beta_*\Phi)$ and uniqueness of $\nu_{\mathrm{eq}}$ can be assured by taking sufficiently nice $\Phi$. Admissibility of $\omega$ is a physical constraint.
- Minimal physicality requirement $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ has to be established for a specific model. (Maybe it can be proven for sufficiently nice $\Phi$ or $\omega$?) Either it holds or the situation is unphysical.
- Regularity of $(\omega_+, \Psi_+)$ follows from the R-Conjecture or it can be established for a specific model.
- Conditions assuring $E_{\Psi_+} \in \mathfrak{C}$ will be discussed tomorrow.

# Approach to Thermal Equilibrium – proof

(2) $\Rightarrow$ (1) We get $s(\omega_+) \geq s(\nu_{\mathrm{eq}})$ as follows:

$$\begin{aligned}
s(\nu_{\mathrm{eq}}|\omega_+) &= -s(\nu_{\mathrm{eq}}) + \nu_{\mathrm{eq}}(E_{\Psi_+}) + P(\Psi_+) && \text{(regularity of } (\omega_+, \Psi_+)\text{)} \\
&= -s(\nu_{\mathrm{eq}}) + \omega(E_{\Psi_+}) \ + \ P(\Psi_+) && \text{(admissibility)} \\
&= -s(\nu_{\mathrm{eq}}) + \omega_+(E_{\Psi_+}) + P(\Psi_+) && (E_{\Psi_+} \in \mathfrak{C}) \\
&= -s(\nu_{\mathrm{eq}}) + s(\omega_+) \geq 0 && (\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+))
\end{aligned}$$

The opposite inequality $s(\omega_+) \leq s(\nu_{\mathrm{eq}})$ in proven analogously

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The opposite inequality $s(\omega_+) \leq s(\nu_{\mathrm{eq}})$ in proven analogously

$$\begin{aligned} s(\omega_+|\nu_{\mathrm{eq}}) &= -s(\omega_+) + \beta_*\omega_+(E_\Phi) + P(\beta_*\Phi) && \text{(regularity of } (\nu_{\mathrm{eq}}, \beta_*\Phi)\text{)} \\ &= -s(\omega_+) + \beta_*\nu_{\mathrm{eq}}(E_\Phi) + P(\beta_*\Phi) && (E_\Phi \in \mathfrak{C} \text{ and admissibility}) \\ &= -s(\omega_+) + s(\nu_{\mathrm{eq}}) \geq 0 && (\nu_{\mathrm{eq}} \in \mathcal{S}_{\mathrm{eq}}(\beta_*\Phi)) \end{aligned}$$

So we have $s(\omega_+) = s(\nu_{\mathrm{eq}})$ and $\omega_+(E_\Phi) = \nu_{\mathrm{eq}}(E_\Phi)$. Gibbs var. principle gives $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\beta_*\Phi) = \{\nu_{\mathrm{eq}}\}$

# Approach to Thermal Equilibrium – admissibility

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Recall we assume the basic conservation laws, in particular $E_\Phi \in \mathfrak{C}$.

**Theorem** **Jakšić, Pillet, S, Tauber '25**

Assume that $\mathcal{S}_{\mathrm{eq}}(\beta_*\Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_*\Phi)$ regular, and that $\omega$ is admissible for $\nu_{\mathrm{eq}}$.
For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

(1) $\omega_+ = \nu_{\mathrm{eq}}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that $E_{\Psi_+} \in \mathfrak{C}$ and $(\omega_+, \Psi_+)$ is regular.

- The admissibility of $\omega$ with respect to $\nu_{\mathrm{eq}}$ is a physical constraint.
- Now we will discuss the case when $\omega$ is **not** admissible. Approach to Thermal Equilibrium cannot happen
- **Principle of maximum entropy:**
  *In the long time the system settles in a state that maximizes entropy while respecting constants of motion.*

## Ruelle's physical equivalence

### Equivalence of observables

Let $A, B \in \mathcal{U}_{\mathrm{real}}$.

$$A \sim_{\mathrm{O}} B \iff \exists c \in \mathbb{R} \;\; \forall \omega \in \mathcal{S}_{\mathrm{I}} \;\; \omega(A) = \omega(B) + c$$

$$A \sim_{\mathrm{sO}} B \iff \forall \omega \in \mathcal{S}_{\mathrm{I}} \;\; \omega(A) = \omega(B)$$

### Equivalence of interactions

Let $\Psi, \Phi \in \mathcal{B}_{\mathrm{b}}$.

$$\Psi \sim_{\mathrm{I}} \Phi \in \mathcal{B}_{\mathrm{b}} \iff E_\Psi \sim_{\mathrm{O}} E_\Phi$$

$$\Psi \sim_{\mathrm{sI}} \Phi \in \mathcal{B}_{\mathrm{b}} \iff E_\Psi \sim_{\mathrm{sO}} E_\Phi$$

### Ruelle's maps

The following maps are isometries:

$$\mathcal{B}_{\mathrm{b}}/{\sim_{\mathrm{I}}} \ni [\Psi] \longmapsto [E_\Psi] \in \mathcal{U}_{\mathrm{real}}/{\sim_{\mathrm{O}}}$$

$$\mathcal{B}_{\mathrm{b}}/{\sim_{\mathrm{sI}}} \ni [\Psi] \longmapsto [E_\Psi] \in \mathcal{U}_{\mathrm{real}}/{\sim_{\mathrm{sO}}}$$

What we will see most often: $C \in \mathfrak{C} \longrightarrow \Psi_C \in \mathcal{B}_{\mathrm{b}} \longrightarrow E_{\Psi_C}$. Then $E_{\Psi_C} \sim_{\mathrm{sO}} C$, i.e. $\omega(C) = \omega(E_{\Psi_C}) \;\; \forall \omega \in \mathcal{S}_{\mathrm{I}}$

# Regular and Physical Constants of Motion

We distinguish two special classes of constants of motion. Let $C \in \mathfrak{C}$.

**Regular constants of motion**

- $C \in \mathfrak{C}_{\mathrm{reg}} \iff (\rho, \Psi_C)$ regular for every $\rho \in \mathcal{S}_{\mathrm{eq}}(\Psi_C)$
- Equilibria of $\mathfrak{C}_{\mathrm{reg}}$: $\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}}) \iff \exists C \in \mathfrak{C}_{\mathrm{reg}} \;\; \rho \in \mathcal{S}_{\mathrm{eq}}(\Psi_C)$

**Physical constants of motion**

- $C \in \mathfrak{C}_{\mathrm{phys}} \iff \Psi_C \in \mathcal{B}_{\mathrm{sd}}$
- Equilibria of $\mathfrak{C}_{\mathrm{phys}}$:

$$\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}}) \iff \exists C \in \mathfrak{C}_{\mathrm{phys}} \;\; \rho \in \mathcal{S}_{\mathrm{eq}}(\Psi_C) \iff \exists \Psi \in \mathcal{B}_{\mathrm{sd}} \;\; \rho \in \mathcal{S}_{\mathrm{eq}}(\Psi) \text{ and } E_\Psi \in \mathfrak{C}$$

- Under R-conjecture: $\mathfrak{C}_{\mathrm{phys}} \subset \mathfrak{C}_{\mathrm{reg}}$
- Note that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$ is equivalent to the condition we need for ATE

**Theorem**

Let $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$. Then

$$s(\omega_+) \leq s(\omega) + \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})} s(\omega|\rho)$$

and

$$\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}}) \quad \Longleftrightarrow \quad \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})} s(\omega|\rho) = s(\omega|\omega_+) = s(\omega_+) - s(\omega)$$

**Proof.** Let $C \in \mathfrak{C}_{\mathrm{reg}}$ be associated to $\Psi_C$. Recall $C \sim_{\mathrm{sO}} E_{\Psi_C}$. For every $\rho \in \mathcal{S}_{\mathrm{eq}}(\Psi_C)$ and $T > 0$

$$\begin{aligned}
s(\omega|\rho) &= -s(\omega) + \omega(E_{\Psi_C}) \ + P(\Psi_C) && (\text{regularity of } (\rho, \Psi_C)) \\
&= -s(\omega) + \omega_T(E_{\Psi_C}) + P(\Psi_C) && (C \in \mathfrak{C}) \\
&= -s(\omega) + \omega_+(E_{\Psi_C}) + P(\Psi_C) && (T_n \uparrow \infty \text{ so that } \omega_{T_n} \to \omega_+) \\
&\geq -s(\omega) + s(\omega_+) && (\text{Gibbs variational principle})
\end{aligned}$$

Hence $\inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})} s(\omega|\rho) \geq s(\omega_+) - s(\omega)$. The other claim follows.

Denote the set of minimizers by $\mathcal{S}_{\mathrm{eq},+}(\mathfrak{C}_{\mathrm{reg}}) = \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}}) \cap \mathcal{S}_+(\omega, \Phi)$.

**Proposition**

- Let $\omega_+, \tilde{\omega}_+ \in \mathcal{S}_{\mathrm{eq},+}(\mathfrak{C}_{\mathrm{reg}})$. Then $s(\omega_+) = s(\tilde{\omega}_+)$ and $s(\omega_+|\tilde{\omega}_+) = 0$.
- There exists $C \in \mathfrak{C}_{\mathrm{reg}}$, unique up to equivalence, such that $\mathcal{S}_{\mathrm{eq},+}(\mathfrak{C}_{\mathrm{reg}}) \subset \mathcal{S}_{\mathrm{eq}}(\Psi_C)$.

**Proof**. Equality of specific entropies follows immediately by the previous theorem.

For specific relative entropy, let $C \in \mathfrak{C}_{\mathrm{reg}}$ be such that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_C)$. Then

$$
\begin{aligned}
s(\tilde{\omega}_+|\omega_+) &= -s(\tilde{\omega}_+) + \tilde{\omega}_+(E_{\Psi_C}) + P(\Psi_C) && \text{(regularity of } (\omega_+, \Psi_C))\\
&= -s(\omega) \ + \ \omega(E_{\Psi_C}) \ + \ P(\Psi_C) && \text{(entropy and } C \text{ conserved)}\\
&= -s(\omega_+) + \omega_+(E_{\Psi_C}) + P(\Psi_C) = 0, && \text{(entropy and } C \text{ conserved \& GVP)}
\end{aligned}
$$

This also means that $\tilde{\omega}_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_C)$, which yields the second claim.

**Constrained Approach to Equilibrium**

The pair $(\omega, \Phi)$ has the property of the Constrained Approach to Equilibrium if

(1) $\mathcal{S}_+(\omega, \Phi) = \{\omega_+\}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$ and $s(\omega_+) = s(\omega) + \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})} s(\omega|\rho)$ with unique minimizer $\omega_+$.

Interpretation of (2): The system relaxes to a **state of maximal entropy compatible with constant of motions**.

Also, by **Quantum Stein lemma**: Among all states in $\mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$, $\omega_+$ is the **least distinguishabl**e from $\omega$.

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- Assume R-Conjecture. Then $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}}) \subset \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})$, so the variational char. of $\mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})$ holds.
  Thus $s(\omega_+) = s(\omega) + \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})} s(\omega|\rho)$ with minimizer $\omega_+$.
- If Part (1) holds, this minimizer is unique. Part (2) reduces to $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$
- Recall in ATE-definition we require $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) \subset \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$

**Constrained Approach to Equilibrium**

The pair $(\omega, \Phi)$ has the property of the Constrained Approach to Equilibrium if

(1) $\mathcal{S}_+(\omega, \Phi) = \{\omega_+\}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$ and $s(\omega_+) = s(\omega) + \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})} s(\omega|\rho)$ with unique minimizer $\omega_+$.

Interpretation of (2): The system relaxes to a **state of maximal entropy compatible with constant of motions**.

Also, by **Quantum Stein lemma**: Among all states in $\mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$, $\omega_+$ is the **least distinguishable** from $\omega$.

- Assume R-Conjecture. Then $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}}) \subset \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})$, so the variational char. of $\mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{reg}})$ holds. Thus $s(\omega_+) = s(\omega) + \inf_{\rho \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})} s(\omega|\rho)$ with minimizer $\omega_+$.
- If Part (1) holds, this minimizer is unique. Part (2) reduces to $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$
- Recall in ATE-definition we require $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) \subset \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$

Assume as in ATE-theorem that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ and $E_{\Psi_+} \in \mathfrak{C}$. Then $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$.

**Key question now:** what conditions ensure $E_{\Psi_+} \in \mathfrak{C}$?

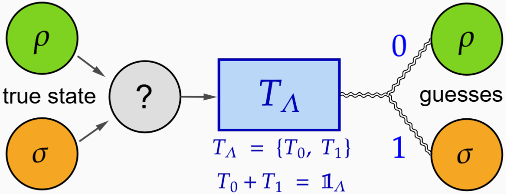
## Quantum Stein Lemma

Let $\rho, \sigma \in \mathcal{S}_{\mathrm{I}}$ and $\Lambda \in \mathcal{F}$.

$H_0$: the system is in state $\rho$

$H_1$: the system is in state $\sigma$

$$\begin{aligned}\text{Type I error } &= \mathrm{P}(\text{we guessed } \rho \mid \text{true state is } \sigma) \\ &= \mathrm{tr}(\sigma_\Lambda T_0) =: \alpha_{\Lambda, T_\Lambda} \\ \text{Type II error } &= \mathrm{P}(\text{we guessed } \sigma \mid \text{true state is } \rho) \\ &= \mathrm{tr}(\rho_\Lambda T_1) =: \beta_{\Lambda, T_\Lambda}\end{aligned}$$




Fix $\epsilon$ small. We pick **optimal $T_\Lambda$ to minimize Type II error while keeping Type I error under control**:

$$\beta_\Lambda = \inf_{T_\Lambda} \{\beta_{\Lambda, T_\Lambda} \mid \alpha_{\Lambda, T_\Lambda} \le \epsilon\}$$

# Quantum Stein Lemma

Let $\rho, \sigma \in \mathcal{S}_{\mathrm{I}}$ and $\Lambda \in \mathcal{F}$.

$H_0$: the system is in state $\rho$

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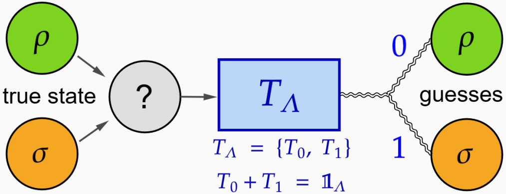
$$\begin{aligned} \text{Type I error } &= \mathrm{P}(\text{we guessed } \rho \mid \text{true state is } \sigma) \\ &= \mathrm{tr}(\sigma_\Lambda T_0) =: \alpha_{\Lambda, T_\Lambda} \\ \text{Type II error } &= \mathrm{P}(\text{we guessed } \sigma \mid \text{true state is } \rho) \\ &= \mathrm{tr}(\rho_\Lambda T_1) =: \beta_{\Lambda, T_\Lambda} \end{aligned}$$




Fix $\epsilon$ small. We pick **optimal $T_\Lambda$ to minimize Type II error while keeping Type I error under control**:

$$\beta_\Lambda = \inf_{T_\Lambda} \{ \beta_{\Lambda, T_\Lambda} \mid \alpha_{\Lambda, T_\Lambda} \leq \epsilon \}$$

$$\beta_\Lambda \approx \mathrm{e}^{-k|\Lambda|} \quad \text{with} \quad k = -\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \beta_\Lambda$$

**Quantum Stein Lemma: $k = s(\rho|\sigma)$ i.e. $\beta_\Lambda \approx \mathrm{e}^{-|\Lambda| s(\rho|\sigma)}$**

QSL gives the **operational meaning of specific relative entropy** as the optimal asymptotic exponential decay rate of Type II error with Type I error fixed (asymmetric hypothesis testing).

# Quantum Stein Lemma

**Theorem**

Assume that for some $\delta > 0$ the following limit exists and is finite for $s \in [0, 1+\delta]$:

$$e(s) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \operatorname{tr}_\Lambda(\sigma_\Lambda^s \rho_\Lambda^{1-s}).$$

Also, assume that the function $e$ is continuous on $[0, 1+\delta]$, differentiable on $(0, 1]$, and $D^+ e(0) < e'(1)$. Then the Quantum Stein Lemma holds for $(\sigma, \rho)$.

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{f}}$ and $\rho \in \mathcal{S}_{\mathrm{eq}}(\Phi)$, $\sigma \in \mathcal{S}_{\mathrm{eq}}(\Psi)$.

- If $d = 1$, then QSL holds for $(\sigma, \rho)$ iff $\Phi, \Psi$ are not equivalent
- If $d > 1$, then QSL holds for $(\sigma, \rho)$ iff $\Phi, \Psi$ are not equivalent and both $\|\Phi\|_{\mathrm{s}}$ and $\|\Psi\|_{\mathrm{s}}$ are small enough

**Interpretation of Constrained Approach to Equilibrium via Quantum Stein Lemma**

Assume the Quantum Stein Lemma holds for all pairs $(\omega, \nu)$, $\nu \in \mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$.
Among all equilibrium states in $\mathcal{S}_{\mathrm{eq}}(\mathfrak{C}_{\mathrm{phys}})$, $\omega_+$ is the **least distinguishable** from $\omega$.

**Key question now:**

Suppose that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$. How can we ensure that $E_{\Psi_+} \in \mathfrak{C}$?

**Next:**

- Structural theory of Constants of Motion:
  - What assures $E_{\Psi_+} \in \mathfrak{C}$? **Conjecture SD** and **Conjecture R+SE**
  - Additional **Conjecture SD+** to characterize $\mathfrak{C}$ (useful for trivial admissibility later on)
- Conservation Laws:
  - **basic CLs** for specific entropy and energy (recall they have been assumed all the time)
  - additional CLs as **partial solutions to Conjectures**

Mini-Dictionary:

- SD stands for *surface-dynamics*
- SD+ for *surface dynamics plus something else*
- R for *regularity*
- SE for *specific (relative) entropy*

## Constants of Motion

- Aim: find natural conditions on $(\Phi, \Psi)$ under which $E_\Psi \in \mathfrak{C}(\alpha_\Phi)$
- Optimally: find characterization of $\mathfrak{C}(\alpha_\Phi)$
- There's going to be two parts:
    1. Conditions related to surface energies and commuting dynamics
    2. Conditions related to regularity and specific relative entropy

# Characterization of $\mathfrak{C}$ via Commuting Dynamics: Motivation

Consider $\omega_+ \in \mathcal{S}_+(\omega, \Phi) \cap \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$. Then

- $\omega_+$ is a KMS state for $s \mapsto \alpha^s_{\Psi_+}$
- $\omega_+$ is $\alpha_\Phi$-invariant, so a KMS state for $s \mapsto \alpha^{-t}_\Phi \circ \alpha^s_{\Psi_+} \circ \alpha^t_\Phi$ for any fixed $t \in \mathbb{R}$.

Shared KMS state $\Rightarrow$ the two dynamics coincide:

$$\alpha^{-t}_\Phi \circ \alpha^s_{\Psi_+} \circ \alpha^t_\Phi = \alpha^s_{\Psi_+} \qquad \forall t, s \in \mathbb{R}$$

That is, $\alpha_{\Psi_+}$ is preserved by $\alpha_\Phi$.

Does $\alpha^t_\Phi \circ \alpha^s_{\Psi_+} = \alpha^s_{\Psi_+} \circ \alpha^t_\Phi$ imply that $E_{\Psi_+} \in \mathfrak{C}$? Are those two equivalent?

# Characterization of $\mathfrak{C}$ via Commuting Dynamics: SD property

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$. Fix $t \in \mathbb{R}$ and for each $\Lambda \in \mathcal{F}$ consider a **dressed Hamiltonian**:

$$H_t(\Lambda) := \mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)} H_\Psi(\Lambda) \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)} = \alpha_{\Phi,\Lambda}^{-t}(H_\Psi(\Lambda)).$$

- Generated dynamics: $s \mapsto \alpha_\Phi^{-t} \circ \alpha_\Psi^s \circ \alpha_\Phi^t$
- Surface energies: $W_t(\Lambda) = \lim_{\Lambda' \uparrow \mathbb{Z}^d} [H_t(\Lambda') - H_t(\Lambda) - H_t(\Lambda' \setminus \Lambda)]$
- The corresponding translation-inv. interaction $\Psi_t$ is uniquely defined as (but it need not be even in $\mathcal{B}_{\mathrm{b}}$!)

$$\Psi_t(\Lambda) = \sum_{X \subset \Lambda} (-1)^{|\Lambda| - |X|} H_t(X)$$

**SD property**

We say $(\Phi, \Psi)$ has the SD property if $W_t(\Lambda)$ exists for all $\Lambda \in \mathcal{F}$, and $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} W_t(\Lambda) = 0$, for $|t| < \epsilon$.

**Theorem** — **Jakšić, Pillet, S, Tauber '25**

Let $(\Phi, \Psi)$ with SD property. Then $\alpha_\Phi^t \circ \alpha_\Psi^s = \alpha_\Psi^s \circ \alpha_\Phi^t \implies E_\Psi \in \mathfrak{C}$.

**Conjecture SD**: For any $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$, the pair $(\Phi, \Psi)$ has SD property.

# Characterization of $\mathfrak{C}$ via Commuting Dynamics: SD+ Property

**Theorem** Jakšić, Pillet, S, Tauber '25

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$. Then $E_\Psi \in \mathfrak{C}$ and $\Psi_t \in \mathcal{B}_{\mathrm{sd}}$ for $|t| < \epsilon \implies \alpha_\Phi^t \circ \alpha_\Psi^s = \alpha_\Psi^s \circ \alpha_\Phi^t$.

## Characterization of $\mathfrak{C}$ via Commuting Dynamics: SD+ Property

**Theorem** — Jakšić, Pillet, S, Tauber '25

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$. Then $E_\Psi \in \mathfrak{C}$ and $\Psi_t \in \mathcal{B}_{\mathrm{sd}}$ for $|t| < \epsilon \implies \alpha_\Phi^t \circ \alpha_\Psi^s = \alpha_\Psi^s \circ \alpha_\Phi^t$.

If the pair $(\Phi, \Psi)$ has SD property, for each $|t| < \epsilon$ we can define the derivation $\delta_t$ on $\mathcal{U}_{\mathrm{loc}}$ by

$$\delta_t(A) = i[H_t(\Lambda), A] + i[W_t(\Lambda), A], \qquad A \in \mathcal{U}_\Lambda.$$

**SD+ property**

We say $(\Phi, \Psi)$ has SD+ property if it has SD property and in addition the following holds for $|t| < \epsilon$:

- $(\mathrm{i} \pm \delta_t)\mathcal{U}_{\mathrm{loc}}$ is dense in $\mathcal{U}$
- $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Psi(\Lambda)) - \alpha_{\Phi,\Lambda}^t(H_\Psi(\Lambda))\| = 0$

**Theorem** — Jakšić, Pillet, S, Tauber '25

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$ with SD+ property. Then $E_\Psi \in \mathfrak{C} \iff \alpha_\Phi^t \circ \alpha_\Psi^s = \alpha_\Psi^s \circ \alpha_\Phi^t$.

**Conjecture SD+**: For any $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$, the pair $(\Phi, \Psi)$ has SD+ property.

# Characterization of $\mathfrak{C}$ via relative entropy

Fix $\Phi \in \mathcal{B}_{\rm sd}$. Recall we assume conservation of specific entropy: $s(\rho \circ \alpha_\Phi^t) = s(\rho)$ for all $\rho \in \mathcal{S}_{\rm I}$, $t \in \mathbb{R}$.

**Proposition**

Let $\Psi \in \mathcal{B}_{\rm sd}$ and $\omega \in \mathcal{S}_{\rm eq}(\Psi)$. Assume that $(\omega, \Psi)$ is regular. Then

$$\forall t \in \mathbb{R} \;\; \forall \rho \in \mathcal{S}_{\rm I} \quad s(\rho \circ \alpha_\Phi^t | \omega) = s(\rho|\omega) \quad \Longleftrightarrow \quad E_\Psi \in \mathfrak{C}$$

Proof is immediate:

$$s(\rho|\omega) = -s(\rho) \quad + \quad \rho(E_\Psi) \quad + \quad P(\Psi)$$

$$s(\rho \circ \alpha_\Phi^t|\omega) = -s(\rho \circ \alpha_\Phi^t) + (\rho \circ \alpha_\Phi^t)(E_\Psi) + P(\Psi)$$

**Conjecture R+SE**: Conjecture R holds and for any $\Phi, \Psi \in \mathcal{B}_{\rm sd}$ and $\omega \in \mathcal{S}_{\rm eq}(\Psi)$ we have

$$\forall t \in \mathbb{R} \;\; \forall \rho \in \mathcal{S}_{\rm I} \quad s(\rho \circ \alpha_\Phi^t|\omega) = s(\rho|\omega)$$

# Characterization of $\mathfrak{C}$ via relative entropy

Fix $\Phi \in \mathcal{B}_{\mathrm{sd}}$. Recall we assume conservation of specific entropy: $s(\rho \circ \alpha_\Phi^t) = s(\rho)$ for all $\rho \in \mathcal{S}_{\mathrm{I}}$, $t \in \mathbb{R}$.

**Proposition**

Let $\Psi \in \mathcal{B}_{\mathrm{sd}}$ and $\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi)$. Assume that $(\omega, \Psi)$ is regular. Then

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Proof is immediate:

$$s(\rho|\omega) = -s(\rho) \quad + \quad \rho(E_\Psi) \quad + \quad P(\Psi)$$

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**Conjecture R+SE**: Conjecture R holds and for any $\Phi, \Psi \in \mathcal{B}_{\mathrm{sd}}$ and $\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi)$ we have

$$\forall t \in \mathbb{R} \;\; \forall \rho \in \mathcal{S}_{\mathrm{I}} \quad s(\rho \circ \alpha_\Phi^t|\omega) = s(\rho|\omega)$$

- We have seen various properties of $(\Phi, \Psi)$ that assure $E_\Psi \in \mathfrak{C}(\alpha_\Phi)$
- Conjectures say: *All reasonable/physical ( $= \mathcal{B}_{\mathrm{sd}}$ ) interactions have these properties*
- Conjectures concern structural properties of QSS, in particular they have nothing to do with time $\to \infty$

# Back to Approach to Thermal Equilibrium

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$.

**Theorem** **Jakšić, Pillet, S, Tauber '25**

Assume that $\mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_* \Phi)$ regular, and that $\omega$ is admissible for $\nu_{\mathrm{eq}}$.
For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

(1) $\omega_+ = \nu_{\mathrm{eq}}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that $(\omega_+, \Psi_+)$ is regular, and $E_{\Psi_+} \in \mathfrak{C}$.

- Recall $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ is a minimal physicality requirement.
- If Conjecture R holds, then $(\omega_+, \Psi_+)$ is regular.
- If either Conjecture SD or Conjecture R+SE holds, then $E_{\Psi_+} \in \mathfrak{C}$.

**Thus, assuming the conjectures and a physically relevant setting, Approach to Thermal Equilibrium follows!**

Key questions now:

- When do these conjectures hold (if at all...)?
- What about SD+?

# (Spoiler) Why SD+? Non-integrable systems

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$.

**Theorem** **Jakšić, Pillet, S, Tauber '25**

Assume $\mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_* \Phi)$ regular, and $E_\Phi$ is the **unique constant of motion** (up to $\sim$).

Let $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ and assume that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ and $E_{\Psi_+} \in \mathfrak{C}$.

Then $\omega_+ = \nu_{\mathrm{eq}}$.

- Regularity of $(\omega_+, \Psi_+)$ is not required! Price to pay: we must verify there are no additional constants of motion. This is where the characterization of $\mathfrak{C}$ via Property SD+ intervenes.
- There's new results started by [Shiraishi'19] proving the non-integrability ($\approx$ unique constant of motion) of a large class of 1D-models, also extended recently to higher dimensions.
- Using SD+ property, we can connect our setting with these results. More on Friday.

## Conservation Laws
## (& partial solutions to Conjectures)

- Basic conservation laws: specific entropy and energy
- Conjectures via conservation laws for various properties

## Basic Conservation Laws

$$\mathcal{B}^r = \{\Phi \,:\, \|\Phi\|_r < \infty\} \quad \text{with } \|\Phi\|_r = \sum_{X \ni 0} \mathrm{e}^{r(|X|-1)} \|\Phi(X)\| \quad \text{and } r > 0$$

$$\mathcal{B}^{\mathrm{diam}}_\gamma = \{\Phi \,:\, \|\Phi\|^{\mathrm{diam}}_\gamma < \infty\} \text{ with } \|\Phi\|^{\mathrm{diam}}_\gamma = \sum_{X \ni 0} |X| [\mathrm{diam}(X)]^\gamma \|\Phi(X)\| \quad \text{and } \gamma > 0$$

- Both $\mathcal{B}^r$ and $\mathcal{B}^{\mathrm{diam}}_\gamma$ are Banach spaces and $\mathcal{B}_{\mathrm{f}}$ is a dense subset of each
- $\mathcal{B}^r$ with $r > 0$ and $\mathcal{B}^{\mathrm{diam}}_\gamma$ with $\gamma > d$ are subsets of $\mathcal{B}_{\mathrm{sd}}$ ([Bratelli-Robinson] and [Bru-Pedra], respectively)
- $\mathcal{B}^{\mathrm{diam}}_\gamma$ and $\mathcal{B}^r$ are incomparable

**Theorem: Basic Conservation Laws**

Let $\Phi \in \mathcal{B}^r$ with $r > 0$, or $\Phi \in \mathcal{B}^{\mathrm{diam}}_\gamma$ with $\gamma > 2d$. Then

$$s(\omega \circ \alpha^t_\Phi) = s(\omega), \quad \omega \circ \alpha^t_\Phi(E_\Phi) = \omega(E_\phi) \qquad \forall t \in \mathbb{R} \;\; \forall \omega \in \mathcal{S}_{\mathrm{I}}$$

Entropy in $\mathcal{B}^r$ [Lanford-Robinson'68], Energy in $\mathcal{B}^r$ [Jakšić-Pillet-Tauber'24], both in $\mathcal{B}^{\mathrm{diam}}_\gamma$ [Jakšić-Pillet-S-Tauber'25]

- For $\mathcal{B}^r$, the proofs of CL for energy and entropy are completely different
- For $\mathcal{B}^{\mathrm{diam}}_\gamma$, the **Lieb–Robinson bound** provides the common framework for both CLs

# Lieb–Robinson bound

**Theorem** [Nachtergaele-Sims-Young'19]

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > d$, and $0 < \epsilon < \gamma - d$. Then:

- Dynamics $\boldsymbol{\alpha}_{\boldsymbol{\Phi}}$ exists
- **Lieb-Robinson bound:** Let $t \in \mathbb{R}$, $\Lambda_0 \subset \Lambda$, and $A \in \mathcal{U}_{\Lambda_0}$.

$$\exists c_t > 0 \quad \|\alpha_\Phi^t(A) - \alpha_{\Phi,\Lambda}^t(A)\| \leq c_t \|A\| |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-(\gamma - d - \epsilon)}$$

Proving the conservation of specific entropy via the Lieb–Robinson bound has been suggested by Wreszinski:

- W. F. Wreszinski: *Irreversibility, the time arrow and a dynamical proof of the second law of thermodynamics.* Quantum Stud.: Math. Found. 7 (2020)
- W. F. Wreszinski: *The second law of thermodynamics as a deterministic theorem for quantum spin systems.* Rev. Math. Phys. 35 (2023)

# CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 1/3

For $\Lambda \in \mathcal{F}$:

$$(\rho \circ \alpha^t_{\Phi,\Lambda})(H_\Lambda(\Phi)) = \rho(\mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)} H_\Lambda(\Phi) \mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)}) = \rho(H_\Lambda(\Phi)),$$

which gives

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} (\rho \circ \alpha^t_{\Phi,\Lambda})(H_\Lambda(\Phi)) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \rho(H_\Lambda(\Phi)) = \rho(E_\Phi).$$

On the other hand,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} (\rho \circ \alpha^t_{\Phi})(H_\Lambda(\Phi)) = (\rho \circ \alpha^t_{\Phi})(E_\Phi).$$

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall \rho \in \mathcal{S}_{\mathrm{I}} \quad \forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} |(\rho \circ \alpha^t_{\Phi})(H_\Lambda(\Phi)) - (\rho \circ \alpha^t_{\Phi,\Lambda})(H_\Lambda(\Phi))| = 0$$

## CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 1/3

For $\Lambda \in \mathcal{F}$:

$$(\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi)) = \rho(\mathrm{e}^{\mathrm{i}tH_\Lambda(\Phi)} H_\Lambda(\Phi) \mathrm{e}^{-\mathrm{i}tH_\Lambda(\Phi)}) = \rho(H_\Lambda(\Phi)),$$

which gives

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|}(\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi)) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|}\rho(H_\Lambda(\Phi)) = \rho(E_\Phi).$$

On the other hand,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|}(\rho \circ \alpha_{\Phi}^t)(H_\Lambda(\Phi)) = (\rho \circ \alpha_\Phi^t)(E_\Phi).$$

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall \rho \in \mathcal{S}_{\mathrm{I}} \quad \forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \tfrac{1}{|\Lambda|}|(\rho \circ \alpha_\Phi^t)(H_\Lambda(\Phi)) - (\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi))| = 0$$

$$\begin{aligned}
|(\rho \circ \alpha_\Phi^t)(H_\Lambda(\Phi)) - (\rho \circ \alpha_{\Phi,\Lambda}^t)(H_\Lambda(\Phi))| &= \left|\rho\big(\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\big)\right| \\
&\leq \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\|
\end{aligned}$$

$\implies$ It suffices to show $\forall t \in \mathbb{R} \quad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|}\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0$

# CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 2/3

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0.$$

Let $\Lambda$ be a cube and $\Lambda_0$ a sub-cube. Define $\tilde{H}_{\Lambda,\Lambda_0}(\Phi) := H_\Lambda(\Phi) - H_{\Lambda_0}(\Phi)$.

$$\begin{aligned} &\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| \\ &\qquad \leq \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + \|\alpha_\Phi^t(\tilde{H}_{\Lambda,\Lambda_0}(\Phi))\| + \|\alpha_{\Phi,\Lambda}^t(\tilde{H}_{\Lambda,\Lambda_0}(\Phi))\| \end{aligned}$$

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0.$$

Let $\Lambda$ be a cube and $\Lambda_0$ a sub-cube. Define $\tilde{H}_{\Lambda,\Lambda_0}(\Phi) := H_\Lambda(\Phi) - H_{\Lambda_0}(\Phi)$.

$$\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\|$$

$$\leq \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + 2\,\|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\|$$

# CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 2/3

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0.$$

Let $\Lambda$ be a cube and $\Lambda_0$ a sub-cube. Define $\tilde{H}_{\Lambda,\Lambda_0}(\Phi) := H_\Lambda(\Phi) - H_{\Lambda_0}(\Phi)$.

$$\begin{aligned}
&\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| \\
&\qquad \leq \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + \ 2\, \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \\
&\qquad \leq \quad \|H_{\Lambda_0}(\Phi)\|\, c_t\, |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} + \ 2\, \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \qquad \text{(Lieb-Robinson bound)}
\end{aligned}$$

# CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 2/3

**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| = 0.$$

Let $\Lambda$ be a cube and $\Lambda_0$ a sub-cube. Define $\tilde{H}_{\Lambda,\Lambda_0}(\Phi) := H_\Lambda(\Phi) - H_{\Lambda_0}(\Phi)$.

$$\begin{aligned}
&\|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\| \\
&\leq \quad \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + \ 2\, \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \\
&\leq \quad \|H_{\Lambda_0}(\Phi)\|\, c_t\, |\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} + \ 2\, \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \qquad \text{(Lieb-Robinson bound)} \\
&\leq \quad |\Lambda_0|\, \|\Phi\|_{\mathrm{s}}\, c_t |\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} + |\Lambda \setminus \Lambda_0| \|\Phi\|_{\mathrm{s}}
\end{aligned}$$

(because $\|H_{\Lambda_0}(\Phi)\| \leq \sum_{X \cap \Lambda_0 \neq \emptyset} \|\Phi(X)\| \leq |\Lambda_0| \sum_{X \ni 0} \|\Phi(X)\| \leq |\Lambda_0| \|\Phi\|_{\mathrm{s}}$)

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**Proposition**

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$. Then

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$$\frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_\Lambda(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_\Lambda(\Phi))\|$$

$$\leq \frac{1}{|\Lambda|} \|\alpha_\Phi^t(H_{\Lambda_0}(\Phi)) - \alpha_{\Phi,\Lambda}^t(H_{\Lambda_0}(\Phi))\| + \frac{2}{|\Lambda|} \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\|$$

$$\leq \frac{1}{|\Lambda|} \|H_{\Lambda_0}(\Phi)\|\, c_t\, |\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} + \frac{2}{|\Lambda|} \|\tilde{H}_{\Lambda,\Lambda_0}(\Phi)\| \qquad \text{(Lieb-Robinson bound)}$$

$$\leq \underbrace{\frac{|\Lambda_0|}{|\Lambda|}}_{\to 1} \|\Phi\|_s\, c_t\, \underbrace{|\Lambda_0| (1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon}}_{\to 0} + \underbrace{\frac{|\Lambda \setminus \Lambda_0|}{|\Lambda|}}_{\to 0} \|\Phi\|_s$$

(because $\|H_{\Lambda_0}(\Phi)\| \leq \sum_{X \cap \Lambda_0 \neq \emptyset} \|\Phi(X)\| \leq |\Lambda_0| \sum_{X \ni 0} \|\Phi(X)\| \leq |\Lambda_0| \|\Phi\|_s$)

Can we construct families $\Lambda$, $\Lambda_0$ such that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \ldots = 0$?

# CLs for energy in $\mathcal{B}_\gamma^{\mathrm{diam}}$: sketch of the proof 3/3

**Lemma.**

Assume $\gamma > 2d$. Let $\Lambda$ be a family of cubes such that $\Lambda \uparrow \mathbb{Z}^d$ and denote the side of $\Lambda$ by $L_\Lambda$.

There exists $p \in (0,1)$ such that the sub-cubes $\Lambda_0 = (1 - L_\Lambda^{-p})\Lambda$ satisfy

(i) $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{|\Lambda \setminus \Lambda_0|}{|\Lambda|} = 0$,

(ii) $\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} = 0$ for any $0 < \epsilon < \gamma - 2d$.

(ii) Since $|\Lambda_0| = \left(1 - L_\Lambda^{-p}\right)^d L_\Lambda^d$ and $\mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda) = L_\Lambda^{1-p}$, we obtain

$$|\Lambda_0|(1 + \mathrm{dist}(\Lambda_0, \mathbb{Z}^d \setminus \Lambda))^{-\gamma+d+\epsilon} \leq L_\Lambda^{d-(1-p)(\gamma-d-\epsilon)}.$$

It follows that $d < (1-p)(\gamma - d - \epsilon)$ if $0 < p < p_0$ with $p_0 := \frac{\gamma-2d-\epsilon}{\gamma-d-\epsilon}$, so (ii) holds

$\Longrightarrow$ **CL for specific energy holds** for $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$

**Proposition**

Let $\Phi \in \mathcal{B}^{\mathrm{diam}}_{\gamma}$ with $\gamma > 2d$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha^t_{\Phi}(H_\Lambda(\Phi)) - \alpha^t_{\Phi,\Lambda}(H_\Lambda(\Phi))\| = 0.$$

**Precious Proposition**

Let $\Phi \in \mathcal{B}^{\mathrm{diam}}_{\gamma}$ with $\gamma > 2d$ and $\Psi \in \mathcal{B}_{\mathrm{s}}$. Then

$$\forall t \in \mathbb{R} \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha^t_{\Phi}(H_\Lambda(\Psi)) - \alpha^t_{\Phi,\Lambda}(H_\Lambda(\Psi))\| = 0.$$

- As we have just seen, it is needed for CL for specific energy (with $\Psi = \Phi$)
- Recall this is one of the conditions defining the SD+ property
- It also intervenes in the partial solution to Conjecture SD

## Conjecture SD

Let $\Phi, \Psi_0 \in \mathcal{B}_{\mathrm{sd}}$. Fix $t \in \mathbb{R}$ and define for $\Lambda \in \mathcal{F}$

$$H_t(\Lambda) := \mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)} H_{\Psi_0}(\Lambda) \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)}.$$

- Generates dynamics $s \mapsto \alpha_\Phi^{-t} \circ \alpha_{\Psi_0}^s \circ \alpha_\Phi^t$
- The corresponding translation-inv. interaction $\Psi_t$ is uniquely defined but it need not be even in $\mathcal{B}_{\mathrm{b}}$
- But its **pressure and specific energy exist** and can be easily computed

**Partial solution to Conjecture SD** [Jakšić–Pillet–S–Tauber'25]

Let $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ for $\gamma > 2d$ and $\Psi_0 \in \mathcal{B}_{\gamma'}^{\mathrm{diam}}$ for $\gamma' > d$. Then $(\Phi, \Psi_0)$ has SD property for all $t \in \mathbb{R}$, i.e.

$$W_t(\Lambda) \text{ exists for all } \Lambda \in \mathcal{F} \text{ and } \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} W_t(\Lambda) = 0$$

Key tool in the proof is the Lieb-Robinson bound via Precious Proposition

# Conjecture SD+

**Refresher on SD+ property**

We say $(\Phi, \Psi_0)$ has SD+ property if it has SD property and in addition the following holds for $|t| < \epsilon$:

(1) $(\mathrm{i} \pm \delta_t)\mathcal{U}_{\mathrm{loc}}$ is dense in $\mathcal{U}$, where $\delta_t(A) = i[H_t(\Lambda) + W_t(\Lambda), A]$ for $A \in \mathcal{U}_\Lambda$.

(2) $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha^t_\Phi(H_{\Psi_0}(\Lambda)) - \alpha^t_{\Phi,\Lambda}(H_{\Psi_0}(\Lambda))\| = 0$

- SD property and Part (2) hold for $\Phi \in \mathcal{B}^{\mathrm{diam}}_\gamma$ for $\gamma > 2d$ and $\Psi \in \mathcal{B}^{\mathrm{diam}}_{\gamma'}$ for $\gamma > d$.
- Part (1) is **much harder**, we only know it holds for pairs of finite-range interactions:

**Partial solution to Conjecture SD+** [Jakšić–Pillet–S–Tauber'25]

Let $\Phi, \Psi_0 \in \mathcal{B}_{\mathrm{f}}$. Then $(\Phi, \Psi_0)$ has SD+ property (so the characterization of $\mathfrak{C}$ via commuting dynamics holds)

**Proof idea**: We follow Bru-Pedra's proof that $(\mathrm{i} \pm \delta_\Psi)\mathcal{U}_{\mathrm{loc}}$ is dense in $\mathcal{U}$ for $\Psi \in \mathcal{B}^{\mathrm{diam}}_\gamma$ with $\gamma > d$, but we need to generalize the Lieb-Robinson bound to the case of the *composite dynamics* generated by $\delta_t$, i.e. instead of $\|(\alpha^s_\Psi - \alpha^s_{\Psi,\Lambda})(A)\| \leq \dots$ we need $\|(\alpha^s - \alpha^s_\Lambda)(A)\| \leq \dots$ where $\alpha^s = \alpha^{-t}_\Phi \circ \alpha^s_{\Psi_0} \circ \alpha^t_\Phi$

# How to tackle Conjecture R+SE

## CL for relative entropy $\Rightarrow$ Property SE

Let $\Psi_0 \in \mathcal{B}_{\mathrm{sd}}$ and $\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi_0)$ such that $(\omega, \Psi_0)$ regular. Recall that SE property means

$$s(\rho \,|\, \omega) = s(\rho \circ \alpha_\Phi^t \,|\, \omega) \qquad \forall t \in \mathbb{R} \;\; \forall \rho \in \mathcal{S}_{\mathrm{I}}$$

**Assume CL for relative entropy**: $s(\rho \,|\, \omega) = s(\rho \circ \alpha_\Phi^t \,|\, \omega \circ \alpha_\Phi^t) \quad \forall t \in \mathbb{R} \;\; \forall \rho \in \mathcal{S}_{\mathrm{I}}$

If $\omega$ is $\alpha_\Phi$-invariant, we recover SE property

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If $\omega$ is $\alpha_\Phi$-invariant, we recover SE property

### CL for regularity $\Rightarrow$ CL for relative entropy

**Assume CL for regularity**: $(\omega, \Psi_0)$ regular $\Rightarrow$ $(\omega \circ \alpha_\Phi^t, \Psi_t)$ regular

We check that $P_{\Psi_t} = P_{\Psi_0}$ and $E_{\Psi_t} = \alpha_\Phi^{-t}(E_{\Psi_0})$. Thus $(\rho \circ \alpha_\Phi^t)(E_{\Psi_t}) = (\rho \circ \alpha_\Phi^t)(\alpha_\Phi^{-t}(E_{\Psi_0})) = \rho(E_{\Psi_0})$, so

$$\begin{aligned} s(\rho \,|\, \omega) &= -s(\rho) \quad + \quad \rho(E_{\Psi_0}) \quad + \quad P_{\Psi_0} \\ &= -s(\rho \circ \alpha_\Phi^t) + (\rho \circ \alpha_\Phi^t)(E_{\Psi_t}) + P_{\Psi_t} = s(\rho \circ \alpha_\Phi^t \,|\, \omega \circ \alpha_\Phi^t) \end{aligned}$$

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If $\omega$ is $\alpha_\Phi$-invariant, we recover SE property

**CL for regularity $\Rightarrow$ CL for relative entropy**

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CL for regularity $\Rightarrow$ CL for relative entropy $\Rightarrow$ Property SE for $\alpha_\Phi$-inv. $\omega$

# CL for weak Gibbsianity

**Definition of weak Gibbsianity** [Jakšić–Pillet–Tauber'24]

We call $\omega \in \mathcal{S}_{\mathrm{I}}$ **weak Gibbs** for $\Psi \in \mathcal{B}_{\mathrm{b}}$ if there exist a family of constants $C_\Lambda > 0$ such that

$$C_\Lambda^{-1} \frac{\mathrm{e}^{-H_\Lambda(\Psi)}}{\mathrm{tr}\,\mathrm{e}^{-H_\Lambda(\Psi)}} \leq \omega_\Lambda \leq C_\Lambda \frac{\mathrm{e}^{-H_\Lambda(\Psi)}}{\mathrm{tr}\,\mathrm{e}^{-H_\Lambda(\Psi)}} \quad \text{and} \quad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\log C_\Lambda}{|\Lambda|} = 0$$

- We denote $\mathcal{S}_{\mathrm{wg}}(\Psi) :=$ set of weak Gibbs states for $\Psi$
- $\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi) \Rightarrow (\omega, \Psi)$ is regular $\Rightarrow \omega \in \mathcal{S}_{\mathrm{eq}}(\Psi)$

**CL for weak Gibbsianity** [Jakšić–Pillet–S–Tauber'25]

Assume that $\Phi \in \mathcal{B}_{\mathrm{f}}$ and either

(a) $d \geq 1$ and $\Psi_0 \in \mathcal{B}^{3r}$ for some $r > 0$ and such that $\|\Psi_0\|_r < r$, or

(b) $d = 1$ and $\Psi_0 \in \mathcal{B}_{\mathrm{f}}$.

Then $\omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0) \Longrightarrow \omega \circ \alpha_\Phi^t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ for $|t|$ small enough. In Case (b) this holds for all $t \in \mathbb{R}$.

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Recall **Thm:** If $\Psi_0 \in \mathcal{B}^r$ with $\|\Psi_0\|_r < r$, or $\Psi_0 \in \mathcal{B}_{\mathrm{f}}$ in $d = 1$, then

$$\omega \in \mathcal{S}_{\mathrm{eq}}(\Psi_0) \;\Leftrightarrow\; (\omega, \Psi_0) \text{ regular } \;\Leftrightarrow\; \omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0)$$ [Jakšić-Pillet-Tauber'24]

Hence, in this context: CL for weak Gibbsianity = CL for regularity, and Conjecture R holds!

## Partial solution to Conjecture R+SE

CL for weak Gibbsianity = CL for regularity ⇒ CL for relative entropy ⇒ Property SE for $\alpha_\Phi$-inv. $\omega$

**CL for relative entropy** **[Jakšić–Pillet–S–Tauber'25]**

Under the same assumptions as CL for weak Gibbsianity:

$$\forall \rho \in \mathcal{S}_{\mathrm{I}} \qquad s(\rho\,|\,\omega) = s(\rho \circ \alpha_\Phi^t\,|\,\omega \circ \alpha_\Phi^t) \quad \text{for} \quad |t| < T_0 \qquad (\text{in dim one } T_0 = \infty)$$

If in addition $\omega$ is $\alpha_\Phi$-invariant, one can take $T_0 = \infty$. (Because then $\omega \circ \alpha_\Phi^t = \omega \in \mathcal{S}_{\mathrm{wg}}(\Psi_0)$ for all $t \in \mathbb{R}$.)

**Partial solution to Conjecture R+SE** **[Jakšić–Pillet–S–Tauber'25]**

Conjecture R+SE holds under the same assumptions as CL for weak Gibbsianity, provided that $\omega$ is $\alpha_\Phi^t$-inv

**CL for regularity/weak Gibbsianity ⇒ CL for relative entropy ⇒ Conjecture R+SE**

| (small time) | (small time) | (all times for $\alpha_\Phi$-invariant state $\omega$) |
|---|---|---|

Recall we apply R+SE to $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, which is $\alpha_\Phi$-invariant!

# State of the art for Conjectures

**Basic CLs for specific energy and entropy**

Hold for either

(a) $\Phi \in \mathcal{B}^r$ with $r > 0$, or

(b) $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$.

**Conjecture SD**

Holds for $\Phi \in \mathcal{B}_\gamma^{\mathrm{diam}}$ with $\gamma > 2d$ and $\Psi \in \mathcal{B}_{\gamma'}^{\mathrm{diam}}$ with $\gamma' > d$.

**Conjecture SD+**

Holds for $\Phi, \Psi \in \mathcal{B}_{\mathrm{f}}$.

**Conjecture R+SE**

Holds for $\Phi \in \mathcal{B}_{\mathrm{f}}$ and either

(a) $d \geq 1$ and $\Psi \in \mathcal{B}^{3r}$ for some $r > 0$ and such that $\|\Psi\|_r < r$, or

(b) $d = 1$ and $\Psi \in \mathcal{B}_{\mathrm{f}}$.

and when $\omega$ is $\alpha_\Phi$-invariant.

## CL for weak Gibbsianity: proof overview

We work under the assumptions for CL for weak Gibbsianity. We denote $\omega_t := \omega \circ \alpha_\Phi^t$

**Characterization of weak Gibbsianity** [Jakšić–Pillet–Tauber'24]

$$\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t) \iff \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \inf_{\substack{A \in \mathcal{U}_\Lambda \\ A > 0}} \frac{\omega_t(A)}{(\omega_t)_{-W_t(\Lambda)}(A)} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \sup_{\substack{A \in \mathcal{U}_\Lambda \\ A > 0}} \frac{\omega_t(A)}{(\omega_t)_{-W_t(\Lambda)}(A)} = 0$$

**Key bound** [Lenci–Rey-Bellet'05]

For any $A \in \mathcal{U}_\Lambda$ such that $A > 0$

$$\exp(-\|W_t(\Lambda)\| - \|\alpha_{-W_t(\Lambda)}^{\mathrm{i}/2}(W_t(\Lambda))\|) \leq \frac{\omega_t(A)}{(\omega_t)_{-W_t(\Lambda)}(A)} \leq \exp(\|W_t(\Lambda)\| + \|\alpha^{\mathrm{i}/2}(W_t(\Lambda))\|)$$

Recall the SD property holds: $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|W_t(\Lambda)\| = 0 \quad \forall t \in \mathbb{R}$

We need to show $\lim\limits_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha^{\mathrm{i}/2}(W_t(\Lambda))\| = 0$ and $\lim\limits_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\alpha_{-W_t(\Lambda)}^{\mathrm{i}/2}(W_t(\Lambda))\| = 0$ (at least for small time)

**Ruelle's bound ('69)**

Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

For any $z$ in this strip $\|\alpha_\Psi^z(A)\| \leq \|A\| \mathrm{e}^{r|\mathsf{supp} A|} C_{z,\Psi}$ with $C_{z,\Psi} = (1 - \frac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}$.

$$\|\alpha^z(A)\| = \|\alpha_{\Psi_0}^z \circ \alpha_\Phi^t(A)\|$$

$$\alpha_\Phi^t(A) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} \sum_{Y_1,\dots,Y_n \subset \mathbb{Z}^d} [\Phi(Y_n) \dots [\Phi(Y_2), [\Phi(Y_1), A]]]$$

Baker-Campbell-Hausdorff Formula

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$$\|\alpha^z(A)\| = \|\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A)\| \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} \sum_{Y_1,\dots,Y_n \subset \mathbb{Z}^d} \|\alpha^z_{\Psi_0}[\Phi(Y_n)\dots[\Phi(Y_2),[\Phi(Y_1),A]]]\|$$

$$\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \alpha^z_{\Psi_0}[\Phi(Y_n)\dots[\Phi(Y_2),[\Phi(Y_1),A]]]$$

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Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha^s_\Psi(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

For any $z$ in this strip $\|\alpha^z_\Psi(A)\| \leq \|A\| e^{r|\mathrm{supp} A|} C_{z,\Psi}$ with $C_{z,\Psi} = (1 - \frac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}$.

$$\|\alpha^z(A)\| = \|\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A)\| \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n)\\ \text{chained to } A}} \|\alpha^z_{\Psi_0}[\Phi(Y_n)\dots[\Phi(Y_2),[\Phi(Y_1),A]]]\| \qquad \text{/ bound the commutator}$$

$$\leq \|\alpha^z_{\Psi_0}(A)\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n)\\ \text{chained to } A}} \prod_{i=1}^{n} \|\alpha^z_{\Psi_0}(\Phi(Y_i))\|$$

$$\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n)\\ \text{chained to } A}} \alpha^z_{\Psi_0}[\Phi(Y_n)\dots[\Phi(Y_2),[\Phi(Y_1),A]]]$$

Baker-Campbell-Hausdorff Formula

**Ruelle's bound ('69)**

Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha^s_\Psi(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

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$$\|\alpha^z(A)\| = \|\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A)\| \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \|\alpha^z_{\Psi_0}[\Phi(Y_n) \dots [\Phi(Y_2), [\Phi(Y_1), A]]]\| \qquad \text{/ bound the commutator}$$

$$\leq \|\alpha^z_{\Psi_0}(A)\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\alpha^z_{\Psi_0}(\Phi(Y_i))\| \qquad \text{/ use Ruelle's bound for } \Psi_0$$

$$\|\alpha^z_{\Psi_0}(\Phi(Y_i))\| \leq \|\Phi(Y_i)\| \mathrm{e}^{r|Y_i|} C_{z,\Psi_0} \leq \|\Phi(Y_i)\| \mathrm{e}^{r(\text{range } \Phi)^d} C_{z,\Psi_0} \qquad \text{Ruelle's bound \& } \Phi \in \mathcal{B}_{\mathrm{f}}$$

# CL for weak Gibbsianity: Ruelle's bound

**Ruelle's bound ('69)**

Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

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$$\|\alpha^z(A)\| = \|\alpha_{\Psi_0}^z \circ \alpha_\Phi^t(A)\| \leq \sum_{n=0}^{\infty} \frac{|t|^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \|\alpha_{\Psi_0}^z[\Phi(Y_n)\dots[\Phi(Y_2),[\Phi(Y_1),A]]]\| \qquad \text{/ bound the commutator}$$

$$\leq \|\alpha_{\Psi_0}^z(A)\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\alpha_{\Psi_0}^z(\Phi(Y_i))\| \qquad \text{/ use Ruelle's bound for } \Psi_0$$

$$\leq C_{z,\Psi_0} \|A\| \mathrm{e}^{r|\operatorname{supp} A|} \sum_{n=0}^{\infty} \frac{(2|t| \mathrm{e}^{r(\operatorname{range}\Phi)^d} C_{z,\Psi_0})^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\Phi(Y_i)\|$$

$$\|\alpha_{\Psi_0}^z(\Phi(Y_i))\| \leq \|\Phi(Y_i)\| \mathrm{e}^{r|Y_i|} C_{z,\Psi_0} \leq \|\Phi(Y_i)\| \mathrm{e}^{r(\operatorname{range}\Phi)^d} C_{z,\Psi_0} \qquad \text{Ruelle's bound } \& \ \Phi \in \mathcal{B}_{\mathrm{f}}$$

## CL for weak Gibbsianity: Ruelle's bound

**Ruelle's bound ('69)**

Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha^s_\Psi(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

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$$\leq \|\alpha^z_{\Psi_0}(A)\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\alpha^z_{\Psi_0}(\Phi(Y_i))\| \qquad \text{/ use Ruelle's bound for } \Psi_0$$

$$\leq C_{z,\Psi_0} \|A\| \mathrm{e}^{r|\mathrm{supp} A|} \sum_{n=0}^{\infty} \frac{(2|t|\mathrm{e}^{r(\mathrm{range}\,\Phi)^d} C_{z,\Psi_0})^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\Phi(Y_i)\| \qquad \text{/ Ruelle's lemma}$$

$$\forall_{n\geq 1} \qquad \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^{n} \|\Phi(Y_i)\| \leq n! \mathrm{e}^{r|\mathrm{supp} A|} (\tfrac{1}{r}\|\Phi\|_r)^n \qquad \text{Ruelle's lemma}$$

**Ruelle's bound ('69)**

Let $A \in \mathcal{U}_{\mathrm{loc}}$ and $\Psi \in \mathcal{B}^r$. The map $\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A) \in \mathcal{U}$ extends analytically to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$.

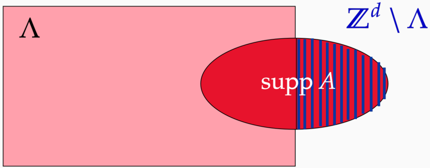
For any $z$ in this strip $\|\alpha_\Psi^z(A)\| \leq \|A\| \mathrm{e}^{r|\mathrm{supp} A|} C_{z,\Psi}$ with $C_{z,\Psi} = (1 - \frac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}$.

$$
\begin{aligned}
\|\alpha^z(A)\| = \|\alpha^z_{\Psi_0} \circ \alpha^t_\Phi(A)\| &\leq \sum_{n=0}^\infty \frac{|t|^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \|\alpha^z_{\Psi_0}[\Phi(Y_n) \dots [\Phi(Y_2), [\Phi(Y_1), A]]]\| && \text{/ bound the commutator} \\
&\leq \|\alpha^z_{\Psi_0}(A)\| \sum_{n=0}^\infty \frac{(2|t|)^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^n \|\alpha^z_{\Psi_0}(\Phi(Y_i))\| && \text{/ use Ruelle's bound for } \Psi_0 \\
&\leq C_{z,\Psi_0} \|A\| \mathrm{e}^{r|\mathrm{supp} A|} \sum_{n=0}^\infty \frac{(2|t| \mathrm{e}^{r(\mathrm{range}\,\Phi)^d} C_{z,\Psi_0})^n}{n!} \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^n \|\Phi(Y_i)\| \\
&\leq C_{z,\Psi_0} \|A\| \mathrm{e}^{2r|\mathrm{supp} A|} (1 - |t|/T_0)^{-1} \quad \text{with} \quad T_0 = \left(\tfrac{2}{r}\|\Phi\|_r \mathrm{e}^{r(\mathrm{range}\,\Phi)^d} C_{z,\Psi_0}\right)^{-1}
\end{aligned}
$$

$$
\forall_{n \geq 1} \qquad \sum_{\substack{(Y_1,\dots,Y_n) \\ \text{chained to } A}} \prod_{i=1}^n \|\Phi(Y_i)\| \leq n! \mathrm{e}^{r|\mathrm{supp} A|} \left(\tfrac{1}{r}\|\Phi\|_r\right)^n \qquad \text{Ruelle's lemma}
$$

# CL for weak Gibbsianity: Ruelle's bound vs. perturbed dynamics

**Perturbed dynamics is much more problematic.** Note that $\alpha^s_{-W_\Lambda(t)}(A) = \alpha^s_{\mathbb{Z}^d \setminus \Lambda} \circ \alpha^s_\Lambda(A)$.




If we applied the previous result:

$$\|\alpha^s_{\mathbb{Z}^d \setminus \Lambda}(\alpha^s_\Lambda(A))\| \leq c_t \|\alpha^s_\Lambda(A)\| \exp(2r|\text{supp}\ \alpha^s_\Lambda(A)|) \leq c_t \|\alpha^s_\Lambda(A)\| \exp(2r|\text{supp}A \cup \Lambda|)$$

Instead of $\exp(|\text{supp}A \cup \Lambda|)$ we should see

$$\exp(|(\text{supp}A \cup \Lambda) \cap (\mathbb{Z}^d \setminus \Lambda)|) \leq \exp(|\text{supp}A|)$$

**Conclusion**

We need a **generalization of Ruelle's bound** for restricted dynamics $\alpha^s_K = \alpha^{-t}_{\Phi|_K} \circ \alpha^s_{\Psi_0|_K} \circ \alpha^t_{\Phi|_K}, \quad K \subset \mathbb{Z}^d$.

# CL for weak Gibbsianity: Ruelle's bound generalized

Let $K \subset \mathbb{Z}^d$ and $A \in \mathcal{U}_{\mathrm{loc}}$. Recall that $\alpha_K^s = \alpha_{\Phi|_K}^{-t} \circ \alpha_{\Psi_{\mathbf{0}}|_K}^s \circ \alpha_{\Phi|_K}^t$.

**Ruelle's bound generalized** [Jakšić–Pillet–S–Tauber'25]

Assume $\Psi \in \mathcal{B}^r$. The map

$$\mathbb{R} \ni s \mapsto \alpha_{\Psi|_K}^s(A) \in \mathcal{U}$$

has an analytic extension to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi\|_r}$. For any $z$ in this strip

$$\|\alpha_{\Psi|_K}^z(A)\| \leq \|A\| \exp(r|\mathrm{supp}A \cap K|) C_{z,\Psi} \quad \text{with} \quad C_{z,\Psi} = (1 - \tfrac{2}{r}\|\Psi\|_r |\operatorname{Im} z|)^{-1}.$$

## CL for weak Gibbsianity: Ruelle's bound generalized

Let $K \subset \mathbb{Z}^d$ and $A \in \mathcal{U}_{\mathrm{loc}}$. Recall that $\alpha_K^s = \alpha_{\Phi|_K}^{-t} \circ \alpha_{\Psi_0|_K}^s \circ \alpha_{\Phi|_K}^t$.

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**Ruelle's bound for composite dynamics** [Jakšić–Pillet–S–Tauber'25]

Assume $\Psi_0 \in \mathcal{B}^{3r}$ such that $\|\Psi_0\|_r < r$ and $\Phi \in \mathcal{B}_{\mathrm{f}}$. Set $R = \operatorname{range} \Phi$ and $T_0 = \left(\frac{2}{r}\|\Phi\|_r C_{z,\Psi_0} \mathrm{e}^{rR^d}\right)^{-1}$.

For all $|t| < T_0$ the map

$$\mathbb{R} \ni s \mapsto \alpha_K^s(A)$$

has an analytic extension to the strip $|\operatorname{Im} z| < \frac{r}{2\|\Psi_0\|_r}$. For any $z$ in this strip

$$\|\alpha_K^z(A)\| \le \|A\| \exp(2r|\mathrm{supp} A \cap K|) C_{z,\Psi_0} (1 - |t|/T_0)^{-1}.$$

Let $\Psi \in \mathcal{B}_{\mathrm{f}}$ and $A \in \mathcal{U}_{\mathrm{loc}}$. Define

$$F_n(x) := \exp\Big((n - R + 1)x + 2\sum_{r=1}^{R} \frac{\exp(rx) - 1}{r}\Big) \quad \text{with} \quad R := \operatorname{range}\Psi$$

**Araki's bound ('69)**

The map $\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A)$ has an analytic extension to the whole complex plane, and for any $z \in \mathbb{C}$

$$\|\alpha_\Psi^z(A)\| \leq F_n(C_\Psi|z|)\|A\|,$$

where $n = \max\{\mathrm{diam}(\mathrm{supp}A), R - 1\}$ and $C_\Psi = 2(R+1)\|\Psi\|_{\mathrm{s}}$

# CL for weak Gibbsianity: Araki's bound in $\dim = 1$

Let $\Psi \in \mathcal{B}_{\mathrm{f}}$ and $A \in \mathcal{U}_{\mathrm{loc}}$. Define

$$F_n(x) := \exp\Big((n - R + 1)x + 2\sum_{r=1}^{R} \frac{\exp(rx) - 1}{r}\Big) \quad \text{with} \quad R := \operatorname{range}\Psi$$

**Araki's bound ('69)**

The map $\mathbb{R} \ni s \mapsto \alpha_\Psi^s(A)$ has an analytic extension to the whole complex plane, and for any $z \in \mathbb{C}$

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**Araki's bound generalized** [Jakšić–Pillet–S–Tauber'25]

Let $K \subset \mathbb{Z}$. Then $\mathbb{R} \ni s \mapsto \alpha_{\Psi|_K}^s(A)$ has an analytic extension to the whole complex plane and for any $z \in \mathbb{C}$

$$\|\alpha_{\Psi|_K}^z(A)\| \leq F_n(C_\Psi |z|)\|A\|,$$

where $n = \max\{\mathrm{diam}(\mathrm{supp} A \cap K), R - 1\}$ and $C_\Psi = 2(R+1)\|\Psi\|_{\mathrm{s}}$

## CL for weak Gibbsianity: proof recap

**Characterization & Key bound** [Jakšić–Pillet–Tauber'24] & [Lenci–Rey-Bellet'05]

For any $A \in \mathcal{U}_\Lambda$ such that $A > 0$

$$-\frac{1}{|\Lambda|}\|W_t(\Lambda)\| - \frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}_{-W_t(\Lambda)}(W_t(\Lambda))\| \le \frac{1}{|\Lambda|}\log\frac{\omega_t(A)}{(\omega_t)_{-W_t(\Lambda)}(A)} \le \frac{1}{|\Lambda|}\|W_t(\Lambda)\| + \frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}(W_t(\Lambda))\|$$

Then $\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ if both bounds go to zero as $\Lambda \uparrow \mathbb{Z}^d$.

- We know the SD property holds: $\lim\limits_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\|W_t(\Lambda)\| = 0,$
- Using **Ruelle/Araki generalized bounds for composite dynamics** we get

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}(W_t(\Lambda))\| = 0 \quad \text{and} \quad \lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\|\alpha^{\mathrm{i}/2}_{-W_\Lambda(t)}(W_t(\Lambda))\| = 0.$$

**CL for weak Gibbsianity holds**

$\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Psi_t)$ for

(a) $|t| < T_0$ if $\Psi \in \mathcal{B}^{3r}$ with $\|\Psi\|_r < r$, and $\Phi \in \mathcal{B}_{\mathrm{f}}$ (Ruelle's bound)

(b) $t \in \mathbb{R}$ if $d = 1$ and $\Psi, \Phi \in \mathcal{B}_{\mathrm{f}}$ (Araki's abound)

# Back to Approach to Thermal Equilibrium

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$.

**Theorem** — **Jakšić, Pillet, S, Tauber '25**

Assume that $\mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_* \Phi)$ regular, and that $\omega$ is admissible for $\nu_{\mathrm{eq}}$.
For $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$, the following statements are equivalent:

(1) $\omega_+ = \nu_{\mathrm{eq}}$

(2) $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ such that $(\omega_+, \Psi_+)$ is regular, and $E_{\Psi_+} \in \mathfrak{C}$.

- Recall $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ is a minimal physicality requirement.
- If Conjecture R holds, then $(\omega_+, \Psi_+)$ is regular.
- If either Conjecture SD or Conjecture R+SE holds, then $E_{\Psi_+} \in \mathfrak{C}$.

**Thus, assuming the conjectures and a physically relevant setting, Approach to Thermal Equilibrium follows!**

Key questions now:

- When do these conjectures hold (if at all...)?
- What about SD+?

# ATE in non-integrable systems

# ATE in non-integrable systems

Let $\omega \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_{\mathrm{sd}}$.

**Theorem** **Jakšić, Pillet, S, Tauber '25**

Assume $\mathcal{S}_{\mathrm{eq}}(\beta_* \Phi) = \{\nu_{\mathrm{eq}}\}$ with $(\nu_{\mathrm{eq}}, \beta_* \Phi)$ regular, and $E_\Phi$ is the **unique constant of motion** (up to $\sim$).

Let $\omega_+ \in \mathcal{S}_+(\omega, \Phi)$ and assume that $\omega_+ \in \mathcal{S}_{\mathrm{eq}}(\Psi_+)$ with $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$ and $E_{\Psi_+} \in \mathfrak{C}$. Then $\omega_+ = \nu_{\mathrm{eq}}$.

- Recent results proving that non-integrability, i.e., unique constant of motion, is **generic** for a large class of models: [Shiraishi'19], [Chiba'24], [Yamaguchi, Chiba, Shiraishi'24], [Chiba'25], [Shiraishi, Tasaki'25] & more
- Characterization of $\mathfrak{C}$ via Property SD+ allows us to connect with these results and prove that the **mixed-field Ising chain has a unique constant of motion.**

**Plan of the last talk**

1. JS proof strategy & results for the Ising chain
2. Connecting the two setups & adapting the proof
3. Summary of JS results for other models

# Overview of JS method & results for Ising chain

## JS-Constants of Motion

- Consider $\Lambda_0 = \{1, \cdots, N\}$ with **periodic boundary condition**. For $S \subset \Lambda_0$ set $D(S) = \max_{i,j \in S} |i - j|_{\mathrm{per}}$
- To each site $i \in \Lambda_0$ attach $\mathcal{H}_i = \mathbb{C}^2$, then $\mathcal{H}_{\Lambda_0} = \otimes_{i \in \Lambda_0} \mathcal{H}_i$ and $\mathcal{U}_{\Lambda_0}$ = bounded operators on $\mathcal{H}_{\Lambda_0}$
- $X_i, Y_i, Z_i$ denote the elements of $\mathcal{U}_{\Lambda_0}$ acting as the usual Pauli matrices on site $i$ and as $\mathbb{I}$ elsewhere.

We consider Hamiltonians of the form

$$H = \sum_{i=1}^{N} (J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}) + \sum_{i=1}^{N} (h_X X_i + h_Y Y_i + h_Z Z_i),$$

where $J_X, J_Y, J_Z, h_X, h_Y, h_Z$ are real constants independent of $i$, and $N + 1$ is identified with 1.

**Definition: JS-Constants of Motion**

We say that $Q \in \mathcal{U}_{\Lambda_0}$ is a JS-Constant of Motion (JS-CM) if $[H, Q] = 0$.

Examples: $\mathbb{I}$ and $H$ and all (linear combinations of) spectral projections of $H$.
To get physically relevant results we introduce a **locality requirement** on JS-CM.

# Locality requirement on JS-CM

We denote by $\mathcal{P}_{\Lambda_0}$ the set of all strings of Pauli basis matrices acting on $\Lambda_0$:

$$\mathcal{P}_{\Lambda_0} = \left\{ A = \bigotimes_{i=1}^{N} A_i \quad \text{where} \quad A_i \in \{X_i, Y_i, Z_i, \mathbb{I}\} \right\}$$

$\mathcal{P}_{\Lambda_0}$ is a basis of $\mathcal{U}_{\Lambda_0}$. For $A \in \mathcal{P}_{\Lambda_0}$ we define $\mathrm{supp}A = \{i \in \Lambda_0 \,|\, A_i \neq \mathbb{I}\}$ and

the **diameter/length** of $A$ as $D(A) = D(\mathrm{supp}A)$ (respecting periodicity!)

**$\ell$-supported observable**

For $\ell \in \mathbb{N}$ we say that $Q \in \mathcal{U}_{\Lambda_0}$ is **$\ell$-supported** if $Q = \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A) \leq \ell}} q_A A$, and $q_A \neq 0$ for some $A$ with $D(A) = \ell$.

We are interested in $\ell$-supported JS-CM's. Examples: $H$ is a 2-supported JS-CM, $\mathbb{I}$ is 0-supported.

We call a JS-CM **local** if it is $\ell$-supported with $\ell \leq N/2$.

# JS results for Mixed-field Ising chain

Let $J_Z \neq 0$ and consider the Hamiltonian

$$H = \sum_{i=1}^{N} J_Z Z_i Z_{i+1} + \sum_{i=1}^{N} (h_X X_i + h_Z Z_i).$$

**JS-non-integrability of mixed-field Ising chain** — **Chiba'24**

If $h_X, h_Z \neq 0$, then ***H* is the unique non-trivial local JS-CM**.

That is:

- There exists no $L$-supported JS-CM for $L = 1$ or $3 \leq L \leq N/2$.
- Every 2-supported JS-CM is a linear combination of $H$ and $\mathbb{I}$.

**Completeness of JS results**

If $h_X = 0$ or $h_Z = 0$, then for any $L$ there exists a non-trivial $L$-supported JS-CM.

That is, turning on the other magnetic field kills all these additional local constants of motion!

Similar results hold for other well-known models (1D and beyond). Proof strategy is always the same.

**Setup.** Suppose that $Q \in \mathcal{U}_{\Lambda_0}$ is an $L$-supported JS-CM. We expand it in Pauli basis as

$$Q = \sum_{\ell=0}^{L} \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A)=\ell}} c_A A$$

Plugging in the formulas for $Q$ and $H$,

$$\sum_{\ell=0}^{L+1} \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A)=\ell}} r_A A = [Q, H] = \sum_{\ell=0}^{L} \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A)=\ell}} \sum_{i=1}^{N} c_A \Big( J_Z [A, Z_i Z_{i+1}] + h_X [A, X_i] + h_Z [A, Z_i] \Big)$$

Using $[Q, H] = 0$, i.e.,

$$r_A = 0 \qquad \text{for all } A \in \mathcal{P}_{\Lambda_0}$$

and comparing both sides, we get a system of linear equations for $c_A$.

**Step 1.** If $3 \leq L \leq N/2$, then $c_A = 0$ for all $A$ such that $D(A) = L$, which means $D(Q) \leq L - 1$, contradiction.

**Step 2.** Work out by hand 1-supported and 2-supported JS-CMs to complete the proof.

# JS column notation for commutators

When [A,B] = C, we say that C is **generated** by A and B.

For example, $X_i Y_{i+1} Y_{i+2} Z_{i+3}$ in [Q, H] is generated by

$$
\begin{aligned}
[Y_i Y_{i+1} Y_{i+2} Z_{i+3},\ Z_i\ ] &= \phantom{-}2\mathrm{i} X_i Y_{i+1} Y_{i+2} Z_{i+3} \\
[X_i Y_{i+1} Y_{i+2} Y_{i+3}, X_{i+3}] &= \phantom{-}2\mathrm{i} X_i Y_{i+1} Y_{i+2} Z_{i+3} \\
[X_i Y_{i+1} X_{i+2},\ Z_{i+2} Z_{i+3}] &= -2\mathrm{i} X_i Y_{i+1} Y_{i+2} Z_{i+3}
\end{aligned}
$$

We express these commutators as

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| term from Q | | $Y$ | $Y$ | $Y$ | $Z$ | | $X$ | $Y$ | $Y$ | $Y$ | | $X$ | $Y$ | $X$ | |
| term from H | | $Z$ | | | | | | | | $X$ | | | | $-Z$ | $Z$ |
| term from [Q , H]/(2i) | | $X$ | $Y$ | $Y$ | $Z$ | | $X$ | $Y$ | $Y$ | $Z$ | | $X$ | $Y$ | $Y$ | $Z$ |

These relations allow us to write the following linear equation

$$h_Z c_{YYYZ} + h_X c_{XYYY} - J_Z c_{XYX} + \ldots = r_{XYYZ} = 0$$

# One example of JS reasoning

Let $L \geq 2$ and consider a string $A = A_i^1 \cdots A_{i+L-1}^L \in \mathcal{P}_{\Lambda_0}$ of length $L$ appearing in $Q = \sum_{\ell=0}^{L} \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A)=\ell}} c_A A$

We show $c_A = 0$ for several classes of strings, depending on their endpoints.

**Case 1.** If $(A^1, A^L) = (X, Y)$ then $c_A = 0$. We note that

$$\begin{array}{cccccc} X & A^2 & \cdots & A^{L-1} & Y & \\ & & & & Z & Z \\ \hline X & A^2 & \cdots & A^{L-1} & X & Z \end{array}$$

The generated string $XA^2 \cdots A^{L-1}XZ$ has length $L+1$. Note there is **no other way** to generate it! This gives

$$J_Z c^L_{XA^2 \cdots A^{L-1}Y} = r^{L+1}_{XA^2 \cdots A^{L-1}XZ}.$$

Since $r^{L+1}_{XA^2 \cdots A^{L-1}XZ} = 0$ and $J_Z \neq 0$, we get $c^L_{XA^2 \cdots A^{L-1}Y} = 0$ as claimed.

Analogous proofs for strings of length $L$ with endpoints $(X, X)$, $(Y, Y)$, $(Y, X)$.

# One example of JS reasoning

**Case 2.** Assume $(A^1, A^L) = (Z, X)$. We have

$$\begin{array}{ccccc} Z & A^2 & \cdots & A^{L-1} & X \\ X & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array}$$

**Case 2.** Assume $(A^1, A^L) = (Z, X)$. We have

$$\begin{array}{ccccc} Z & A^2 & \cdots & A^{L-1} & X \\ X & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} X & A^2 & \cdots & A^{L-1} & X \\ -Z & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} Y & A^2 & \cdots & A^{L-1} & Y \\ & & & & Z \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array}$$

- Since the endpoints $YA^2 \cdots A^{L-1}X$ are both non-$Z$, this string cannot arise via the action of $ZZ$.
- However, the action of $X$ or $Z$ inside the string is possible. Here's one example:

$$\begin{array}{ccccc} Y & Z & \cdots & A^{L-1} & X \\ & X & & & \\ \hline Y & Y & \cdots & A^{L-1} & X \end{array}$$

# One example of JS reasoning

**Case 2.** Assume $(A^1, A^L) = (Z, X)$. We have

$$\begin{array}{ccccc} Z & A^2 & \cdots & A^{L-1} & X \\ X & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} X & A^2 & \cdots & A^{L-1} & X \\ -Z & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} Y & A^2 & \cdots & A^{L-1} & Y \\ & & & & Z \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array}$$

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# One example of JS reasoning

**Case 2.** Assume $(A^1, A^L) = (Z, X)$. We have

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$$\begin{array}{ccccc} Y & Z & \cdots & A^{L-1} & X \\ & X & & & \\ \hline Y & Y & \cdots & A^{L-1} & X \end{array}$$

$$h_X c^L_{Z\cdots X} - h_Z \underbrace{c^L_{X\cdots X}}_{=0} + h_Z \underbrace{c^L_{Y\cdots Y}}_{=0} + \overbrace{\sum h_{x/z} \underbrace{c^L_{Y\cdots X}}_{=0}}^{\text{inside action}} = r^L_{Y\cdots X} = 0$$

# One example of JS reasoning

**Case 2.** Assume $(A^1, A^L) = (Z, X)$. We have

$$\begin{array}{ccccc} Z & A^2 & \cdots & A^{L-1} & X \\ X & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} X & A^2 & \cdots & A^{L-1} & X \\ -Z & & & & \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array} \qquad \begin{array}{ccccc} Y & A^2 & \cdots & A^{L-1} & Y \\ & & & & Z \\ \hline Y & A^2 & \cdots & A^{L-1} & X \end{array}$$

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- So strings of length $L$ with endpoints $(Z, X)$ and $(X, Z)$ also vanish. Still remain: $(Z, Z)$, $(Z, Y)$, $(Y, Z)$
- We continue until we show all of them vanish, hence $D(Q) < L$.
- The tricky part is to find the optimal order in which we consider and eliminate various types of $c_A$ as it highly depends on the model. In this regard, we exactly follow [Chiba'24].

# Reduction step
# & connecting the setups

# Reduction step

**We now drop the periodic boundary condition.** For simplicity we consider 1D, but all transfers to higher dim.

**SD+ Property: characterization of $\mathfrak{C}(\alpha_\Phi)$ via commuting dynamics** **Jakšić, Pillet, S, Tauber'25**

Let $\Phi, \Psi \in \mathcal{B}_{\mathrm{f}}$. Then $E_{\Psi_+} \in \mathfrak{C}(\alpha_\Phi)$ iff $\alpha_\Phi^t \circ \alpha_\Psi^s = \alpha_\Psi^s \circ \alpha_\Phi^t$.

**Remark.** [Araki'90] was the first to discuss constants of motion in the setting of infinitely extended quantum spin systems, defining them via $\delta_\Phi \circ \delta_\Psi = \delta_\Psi \circ \delta_\Phi$ on $\mathcal{U}_{\mathrm{loc}}$. Under SD+, the two definitions are equivalent.

We fix the box $\Lambda_0 = \{1, \ldots, N\}$. For some bigger box $\Lambda$ we have

$$[H_\Phi(\Lambda), [H_\Psi(\Lambda), A]] = [H_\Psi(\Lambda), [H_\Phi(\Lambda), A]] \qquad \forall A \in \mathcal{U}_{\Lambda_0}$$

and the Jacobi identity gives

$$[[H_\Psi(\Lambda), H_\Phi(\Lambda)], A] = 0 \qquad \forall A \in \mathcal{U}_{\Lambda_0}$$

Then by standard calculations we obtain

$$\mathrm{tr}_{\Lambda \setminus \Lambda_0}([H_\Psi(\Lambda), H_\Phi(\Lambda)]) = 0.$$

# Reduction step

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$$\mathrm{tr}_{\Lambda\setminus\Lambda_0}([H_\Psi(\Lambda), H_\Phi(\Lambda)]) = 0.$$

and substitute into it

$$H_\Psi(\Lambda) = H_\Psi(\Lambda_0) + H_\Psi(\Lambda \setminus \Lambda_0) + W_\Psi(\Lambda_0),$$

$$H_\Phi(\Lambda) = H_\Phi(\Lambda_0) + H_\Phi(\Lambda \setminus \Lambda_0) + W_\Phi(\Lambda_0)$$

This leads to

$$[H_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] + \mathcal{W} = -\,\mathrm{tr}_{\Lambda\setminus\Lambda_0}([W_\Psi(\Lambda_0), W_\Phi(\Lambda_0)])$$

where

$$\mathcal{W} = \mathrm{tr}_{\Lambda\setminus\Lambda_0}\Big([W_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] + [H_\Psi(\Lambda_0), W_\Phi(\Lambda_0)] + [H_\Psi(\Lambda \setminus \Lambda_0), W_\Phi(\Lambda_0)] + [W_\Psi(\Lambda_0), H_\Phi(\Lambda \setminus \Lambda_0)]\Big).$$

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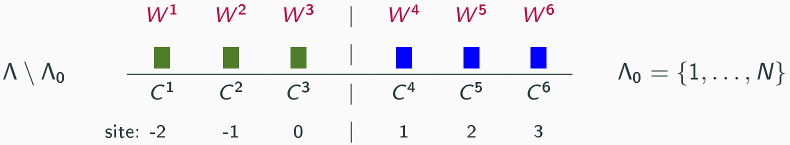
We will now prove that $\mathcal{W} = 0$, so

$$[H_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] = -\,\mathrm{tr}_{\Lambda\setminus\Lambda_0}([W_\Psi(\Lambda_0), W_\Phi(\Lambda_0)]) =: Q'$$

Then we will prove that if $H_\Psi(\Lambda_0)$ is $L$-supported for some $L$, then **$Q'$ is at most $(L-1)$-supported**.

*Intuition: $H_\Psi(\Lambda_0)$ is a candidate for a constant of motion. Recall JS-CM condition:* $[Q, H] = 0$.

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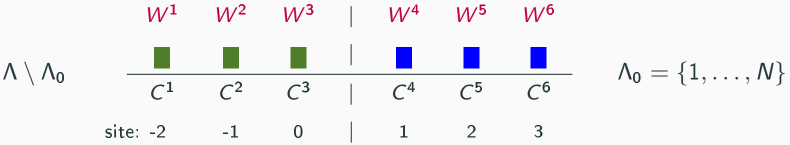
$C := [W, H] = [$ Pauli string $W$ stretching across the boundary of $\Lambda_0$, Pauli string $H$ fully inside or outside $\Lambda_0]$

When $\mathrm{tr}_{\Lambda \setminus \Lambda_0} C \neq 0$?

# Proof that $\mathcal{W} = 0$

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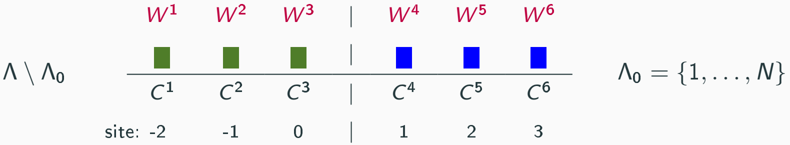


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$C := [W, H] = [$ Pauli string $W$ stretching across the boundary of $\Lambda_0$, Pauli string $H$ fully inside or outside $\Lambda_0]$

When $\mathrm{tr}_{\Lambda\setminus\Lambda_0} C \neq 0$?

- $C = [W, H] \neq 0$ only if there is an **odd number of Pauli matrices flipped to other Pauli matrices**.
- Assume $C \neq 0$. Then $\mathrm{tr}_{\Lambda\setminus\Lambda_0}(C^1C^2C^3|C^4C^5C^6) = \mathrm{tr}(C^1C^2C^3)C^4C^5C^6$
- $\mathrm{tr}(C^1C^2C^3) \neq 0$ only if $C^1 = C^2 = C^3 = \mathbb{I}$

So $\mathrm{tr}_{\Lambda\setminus\Lambda_0} C \neq 0$ only if Pauli matrices **both inside and outside** $\Lambda_0$ are affected.

# Proof that $\mathcal{W} = 0$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $W^1$ | $W^2$ | $W^3$ | \| | $W^4$ | $W^5$ | $W^6$ |
| $\Lambda \setminus \Lambda_0$ | | $C^1$ | $C^2$ | $C^3$ | \| | $C^4$ | $C^5$ | $C^6$ |
| | site: | -2 | -1 | 0 | \| | 1 | 2 | 3 |

$\Lambda_0 = \{1, \ldots, N\}$

---

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But $H$ acts either inside or outside, contradiction. Thus $[H_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] = -\mathrm{tr}_{\Lambda\setminus\Lambda_0}([W_\Psi(\Lambda_0), W_\Phi(\Lambda_0)])$

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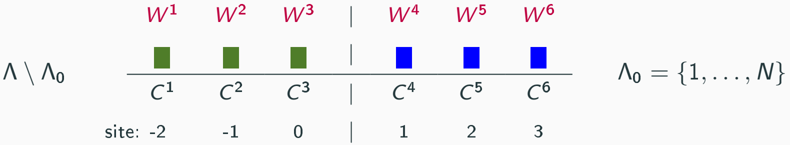


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So $\mathrm{tr}_{\Lambda\setminus\Lambda_0} C \neq 0$ only if Pauli matrices **both inside and outside** $\Lambda_0$ are affected.

***Note these conditions hold for every string $C$ given by some commutator!***

But $H$ acts either inside or outside, contradiction. Thus $[H_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] = -\mathrm{tr}_{\Lambda\setminus\Lambda_0}([W_\Psi(\Lambda_0), W_\Phi(\Lambda_0)])$

# Proof that $D(Q') \leq L - 1$

$$\Lambda \setminus \Lambda_0 \qquad \begin{array}{ccc|ccc} W^1 & W^2 & W^3 & W^4 & W^5 & W^6 \\ \widetilde{W}^1 & \widetilde{W}^2 & \widetilde{W}^3 & \widetilde{W}^4 & \widetilde{W}^5 & \widetilde{W}^6 \\ \hline C^1 & C^2 & C^3 & C^4 & C^5 & C^6 \end{array} \qquad \Lambda_0 = \{1, \ldots, N\}$$

---

Suppose $H_\Psi(\Lambda)$ is $L$-supported for some $L \in \mathbb{N}$. We claim that

$$D(Q') \leq L - 1 \qquad \text{for} \qquad Q' := \mathrm{tr}_{\Lambda \setminus \Lambda_0}([W_\Phi(\Lambda_0), W_\Psi(\Lambda_0)])$$

Consider

$$C := [W, \widetilde{W}] = [\text{ Pauli string } W \text{ across the boundary of } \Lambda_0, \text{ Pauli string } \widetilde{W} \text{ across the boundary of } \Lambda_0]$$

Recall that $\mathrm{tr}_{\Lambda \setminus \Lambda_0} C \neq 0$ only if $C = \mathbb{I} \cdots \mathbb{I} \,|\, C^i C^{i+1} \cdots C^L$ with $i \geq 2$.

## Proof that $D(Q') \leq L - 1$

$$\begin{array}{cccccccc} & W^1 & W^2 & W^3 & | & W^4 & W^5 & W^6 \\ \Lambda \setminus \Lambda_0 & \widetilde{W}^1 & \widetilde{W}^2 & \widetilde{W}^3 & | & \widetilde{W}^4 & \widetilde{W}^5 & \widetilde{W}^6 \\ \hline & C^1 & C^2 & C^3 & | & C^4 & C^5 & C^6 \end{array} \qquad \Lambda_0 = \{1, \ldots, N\}$$

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Suppose $H_\Psi(\Lambda)$ is $L$-supported for some $L \in \mathbb{N}$. We claim that

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Recall that $\mathrm{tr}_{\Lambda \setminus \Lambda_0} C \neq 0$ only if $C = \mathbb{I} \cdots \mathbb{I} \,|\, C^i C^{i+1} \cdots C^L$ with $i \geq 2$.

In consequence, $\quad \mathrm{tr}_{\Lambda \setminus \Lambda_0}(C) = C^i C^{i+1} \cdots C^L$ is supported inside $\{1, \ldots, L-1\}$

thus it is of the form $A_1 \otimes \cdots \otimes A_{L-1}$

The other endpoint of $\Lambda_0$ analogously. So $\mathrm{tr}_{\Lambda \setminus \Lambda_0}(C)$ is a lin. comb. of $A_1 \cdots A_{L-1}$ and $A_{N-L+2} \cdots A_N$

# Recovering the JS proof structure

Recall $\Psi \in \mathcal{B}_{\mathrm{f}}$ is a candidate for a Constant of Motion. We know that $H_\Psi(\Lambda)$ is $L$-supported for $L \in \mathbb{N}$.

Assume $\Phi \in \mathcal{B}_{\mathrm{f}}$ generates

$$H_\Phi(\Lambda_0) = \sum_{i=1}^{N-1}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}) + \sum_{i=1}^{N}(h_X X_i + h_Y Y_i + h_Z Z_i),$$

$$\xRightarrow{\text{Reduction step}} \quad [H_\Psi(\Lambda_0), H_\Phi(\Lambda_0)] \; = \; \underbrace{-\operatorname{tr}_{\Lambda_0^c}([W_\Psi(\Lambda_0), W_\Phi(\Lambda_0)])}_{Q'} \overset{\text{basis exp.}}{=} \sum_{\ell=0}^{L+1} \sum_{\substack{A \in \mathcal{P}_{\Lambda_0} \\ D(A)=\ell}} r_A A$$

AND we know that $D(Q') \leq L-1$, which means $r_A = 0$ if $D(A) \in \{L, L+1\}$.

Recall in JS setup we have $r_A = 0$ for **all** $A$. Here we have less, **but** we also know the structure of $Q'$:

**Crucial properties of $Q'$**

$D(Q') \leq L-1$ and $Q'$ is a linear comb. of the boundary-touching strings $A_1 \cdots A_{L-1}$ and $A_{N-L+2} \cdots A_N$

# Mixed-field Ising chain is trivially admissible

**Theorem.** Jakšić, Pillet, S, Tauber'25

Let $\Phi \in \mathcal{B}_{\mathrm{f}}$ generate $H_\Phi(\Lambda_0) = \sum_{i=1}^{N-1} J_Z Z_i Z_{i+1} + \sum_{x=1}^{N} (h_X X_i + h_Z Z_i)$ with $h_X, h_Z \neq 0$. Then $\mathfrak{C}(\alpha_\Phi) = \{E_\Phi\}$.

**Step 1.** **(As in JS, be careful about the boundary)**

Assume $3 \leq L \leq \frac{N}{2}$. Using $r_A = 0$ if $D(A) \in \{L, L+1\}$, we show $c_A = 0$ for every string $A$ of length $L$; hence, $D(H_\Psi(\Lambda_0)) \leq L - 1$. This contradicts the assumption that $H_\Psi(\Lambda_0)$ is $L$-supported.

*Miraculously, the JS proof only uses $r_A = 0$ for $A$ of length $L$ and $L+1$, so we can follow it.*
*We just need some technical tweaks due to boundary.*

**Step 2.** **(Extra knowledge about Q' needed, then as in JS)**

We investigate by hand the cases $L = 1$ and $L = 2$, showing that $H_\Psi(\Lambda_0) = H_\Phi(\Lambda_0)$ (up to equivalence)

*Here the JS proof uses the full strength of its assumptions, while we have less. But we can make up for it by exploiting the properties of $Q'$ derived earlier, and then we can again follow JS.*

# Proof overview: Step 1

**Step 1.**

We assume $3 \leq L \leq N/2$ and for a string $A$ of length $L$ show:

(i) $c_A = 0$ if $A$ is **not** of the form $Z \cdots Y$ or $Y \cdots Z$ or $Z \cdots Z$.

(ii) $c_A = 0$ if $A$ is the form $Z \cdots Y$ or $Y \cdots Z$.

(iii) $c_A = 0$ if $A$ is the form $Z \cdots Z$.

This contradicts the assumption that $H_\Psi(\Lambda_0)$ is $L$-supported. Hence $H_\Psi(\Lambda_0)$ is at most 2-supported

We follow **very closely** the JS proof but we have to be **careful about the boundary**:

- Some equations do not hold for the boundary-touching strings as there is no $Z_0 Z_1$ nor $Z_N Z_{N+1}$ in $H_\Phi(\Lambda_0)$
- **Shifting procedure**: $$c_{A_i} = c_{A'_{i+1}} = c_{A''_{i+2}} = \ldots = c_{\hat{A}_{i+k}} = 0$$

  In JS setup, periodicity allows an unlimited number of shifts to the right.

  The boundary forces us to make sure that:

  - The shifting procedure works in **both directions**
  - There is **sufficient space** (either to the left or to the right) for the required number of shifts. This is where the assumption $L \leq N/2$ comes from

**Step 2.**

We investigate by hand $L = 1$ and $L = 2$ and show that $H_\Psi(\Lambda_0) = \alpha H_\Phi(\Lambda_0) + \beta\mathbb{I}$

**Step 2a.** Assume $L = 2$. Recall $Q'$ is a linear combination of $A_1$ and $A_N$:

$$[H_\Phi(\Lambda_0), H_\Psi(\Lambda_0)] = Q' = (r_{X_1}X_1 + r_{Y_1}Y_1 + r_{Z_1}Z_1) + (r_{X_N}X_N + r_{Y_N}Y_N + r_{Z_N}Z_N) + r_I I$$

We must show that $r_{Z_1} = r_{Z_N} = 0$. (Recall in JS we have $Q' = 0$ automatically)

Recall that

$$Q' := -\operatorname{tr}_{\Lambda_0^c}([W_\Phi(\Lambda_0), W_\Psi(\Lambda_0)]) \quad \text{and} \quad W_\Phi(\Lambda_0) = -\tfrac{1}{2}J_Z(Z_0Z_1 + Z_NZ_{N+1})$$

Note **there does not exist** $A_0, A_1 \in \{X, Y, Z\}$ that satisfy the following commutation relation:

$$\begin{array}{c|c} A_0 & A_1 \\ Z_0 & Z_1 \\ \hline \mathbb{I}_0 & Z_1 \end{array} \qquad \text{where the structure is} \qquad \begin{array}{c} \text{term from } W_\Psi(\Lambda_0) \\ \text{term from } W_\Phi(\Lambda_0) \\ \hline \text{term from Q'/(2i)} \end{array}$$

in consequence, $r_{Z_1} = 0 = r_{Z_N}$. Knowing that $r_{Z_j} = 0$ for all $j$, we can **follow JS proof again.**

## Proof overview: Step 2

**Step 2b.** First, show that the **only potentially non-zero coefficients** of $H_\Psi(\Lambda_0)$ are $c_{Z_j Z}$, $c_{Z_j}$, $c_{X_j}$, and $c_{\mathbb{I}}$.

We already know that for every $j$

$$c_{X_j X} = c_{X_j Y} = c_{Y_j X} = c_{Y_j Y} = c_{X_j Z} = c_{Z_j X} = 0$$

To show that $c_{Z_j Y} = c_{Y_j Z} = c_{Y_j} = 0$, we use Step 2a that guarantees $r_{Z_j} = 0$ for **every** $j \in \Lambda_0$.

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Namely

$$[Y_j, -X_j] = (2i)\, Z_j \quad \Longrightarrow \quad -h_X c_{Y_j} = r_{Z_j} = 0 \quad \Longrightarrow \quad c_{Y_j} = 0 \quad \forall j \in \Lambda_0$$

and

$$\begin{array}{cc} Y_j & \\ Z_j & Z_{j+1} \\ \hline X_j & Z_{j+1} \end{array} \quad \begin{array}{cc} Y_j & Z_{j+1} \\ Z_j & \\ \hline X_j & Z_{j+1} \end{array} \Rightarrow J_Z c_{Y_j} + h_Z c_{Z_j Y_{j+1}} = 0 \quad \Bigg| \quad \begin{array}{cc} & Y_{j+1} \\ Z_j & Z_{j+1} \\ \hline Z_j & X_{j+1} \end{array} \quad \begin{array}{cc} Z_j & Y_{j+1} \\ & Z_{j+1} \\ \hline Z_j & X_{j+1} \end{array} \Rightarrow J_Z c_{Y_{j+1}} + h_Z c_{Z_j Y_{j+1}} = 0$$

Therefore $\quad c_{Z_j Y_{j+1}} = c_{Y_j Z_{j+1}} = 0 \quad$ for all $1 \le j \le N-1$

**Step 2b.** First, show that the **only potentially non-zero coefficients** of $H_\Psi(\Lambda_0)$ are $c_{Z_jZ}$, $c_{Z_j}$, $c_{X_j}$, and $c_{\mathbb{I}}$.

We already know that for every $j$

$$c_{X_jX} = c_{X_jY} = c_{Y_jX} = c_{Y_jY} = c_{X_jZ} = c_{Z_jX} = 0$$

To show that $c_{Z_jY} = c_{Y_jZ} = c_{Y_j} = 0$, we use Step 2a that guarantees $r_{Z_j} = 0$ for **every** $j \in \Lambda_0$.

Next, we show that the **coefficients are in correct proportion** (this also means $L = 1$ is impossible):

$$c_{Z_jZ}/J_Z = c_{Z_j}/h_Z = c_{X_j}/h_X$$

Indeed,

$$\begin{array}{cc} Z_j & Z_{j+1} \\ X_j & \\ \hline Y_j & Z_{j+1} \end{array} \qquad \begin{array}{cc} X_j & \\ -Z_j & Z_{j+1} \\ \hline Y_j & Z_{j+1} \end{array} \qquad \Longrightarrow \quad h_X c_{Z_jZ} - J_Z c_{X_j} = 0 \quad \text{i.e.} \quad c_{X_j}/h_X = c_{Z_jZ}/J_Z$$

The other one is analogous. Setting $\alpha := c_{\mathbb{I}}$ and $\beta := c_{X_j}/h_X$, we finally obtain

$$H_\Psi(\Lambda_0) = \alpha\mathbb{I} + \beta H_\Phi(\Lambda_0)$$

# Recap for mixed-field Ising chain

**Theorem.** **Jakšić, Pillet, S, Tauber'25**

Let $\Phi \in \mathcal{B}_{\mathrm{f}}$ generate $H_\Phi(\Lambda_0) = \sum_{i=1}^{N-1} J_Z Z_i Z_{i+1} + \sum_{x=1}^{N} (h_X X_i + h_Z Z_i)$ with $h_X, h_Z \neq 0$. Then $\mathfrak{C}(\alpha_\Phi) = \{E_\Phi\}$.

- Main difference wrt JS: definition of Constants of Motion. **JS-CM is stronger than our-CM.** Also, **periodic vs. open** boundary condition, which entails technical tweaks.
- Open boundary condition are discussed in [Chiba'24] under JS-CM.
- Step 1 of JS proof does not use the full strength of JS-CM: While assuming $Q' = 0$, it only uses $D(Q') \leq L - 1$ (which is exactly what our-CM provides).
- Step 2 of JS proof uses the full strength of JS-CM: We catch up by using the explicit formula for $Q' = -\operatorname{tr}_{\Lambda_0^c}([W_\Phi(\Lambda_0), W_\Psi(\Lambda_0)])$.

# Other models

Back to periodic boundary condition. Let $J_X, J_Y \neq 0$ and consider

$$H_\Phi(\Lambda_0) = \sum_{i=1}^{N}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1}) + \sum_{i=1}^{N}(h_X X_i + h_Y Y_i).$$

- If $h_X = h_Y = 0$, the XY model has **infinitely many** (non-trivial local) JS-CM [Lieb, Schultz, Mattis'61]
- If $h_X \neq 0$ or $h_Y \neq 0$:
  - If $J_Y = -J_X(h_Y/h_X)^2$ and $N$ is even, then there are **two** JS-CMs:

$$H_\Phi(\Lambda_0) \quad \text{and} \quad Q = \sum_{i=1}^{N}(-1)^i(h_X X_i + h_Y Y_i)(h_X X_{i+1} + h_Y Y_{i+1})$$

  - Otherwise, $H_\Phi(\Lambda_0)$ is the **unique** JS-CM

Back to periodic boundary condition. Let $J_X, J_y, J_Z \neq 0$ and consider

$$H_\Phi(\Lambda_0) = \sum_{i=1}^{N}(J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}) + \sum_{i=1}^{N}(h_X X_i + h_Y Y_i + h_Z Z_i).$$

- If $J_X = J_Y = J_Z$, the XXX model has **infinitely many** JS-CMs (for any $h_X, h_Y, h_Z$) [Bethe'31]
- If $J_X = J_Y \neq J_Z$:
  - if $h_X = h_Y = 0$, the XXZ model has **infinitely many** JS-CMs [C.-N. Yang, C.-P. Yang'66]
  - if $h_X \neq 0$ or $h_Y \neq 0$, then $H_\Phi(\Lambda_0)$ is the **unique** JS-CM
- If $J_X, J_Y, J_Z$ are all different:
  - if $h_X = h_Y = h_Z = 0$, the XYZ model has **infinitely many** JS-CMs [Baxter'71]
  - if $J_X = -J_Y$ and $h_Z \neq 0$, there are **two** JS-CMs: $H_\Phi(\Lambda_0)$ and $Q = \sum_{i=1}^{N}(-1)^i h_Z Z_i$
  - if at least one of $h_X, h_Y, h_Z$ is non-zero, $H_\Phi(\Lambda_0)$ is the **unique** JS-CM

In particular, among such chains **there is no missing integrable system that awaits to be discovered**.

# Summary of JS results: other models

**Higher dim Ising model** **Chiba'25**

Let $\Lambda_0 = \{1, \ldots, N\}^d$. Assume $J_Z \neq 0$ and $h_X \neq 0$ and consider

$$H_\Phi(\Lambda_0) = \sum_{\substack{i,j \in \Lambda_0 \\ |i-j|=1}} J_Z Z_i Z_j + \sum_{i \in \Lambda_0} (h_X X_i + h_Z Z_i).$$

Any local JS-CM is a linear combination of $H_\Phi(\Lambda_0)$ and $\mathbb{I}$.

**Higher dim Heisenberg model** **Shiraishi, Tasaki'25**

Assume $J_X \neq 0$ and $J_Y \neq 0$ and consider

$$H_\Phi(\Lambda_0) = \sum_{\substack{i,j \in \Lambda_0 \\ |i-j|=1}} (J_X X_i X_j + J_Y Y_i Y_j + J_Z Z_i Z_j) + \sum_{i \in \Lambda_0} (h_X X_i + h_Y Y_i + h_Z Z_i).$$

Any local JS-CM is a linear combination of $H_\Phi(\Lambda_0)$ and some one-body operator and $\mathbb{I}$.

**Other 1D models considered by JS**

- Heisenberg model with next-nearest-neighbor interaction [Shiraishi'24]
- PXP model [Park, Lee'24]
- Spin-1 model with bilinear biquadratic interactions [Park, Lee'24], [Hokkyo, Yamaguchi, Chiba'24]

# Outlook

- We keep working on getting all the JS results in our setting:
  - So far, only the Ising chain has been written down with all details
  - Next target is the XY chain – JS have not considered open boundary condition for it
- Conjecture: **generically, finite-range interactions have no non-trivial local constants of motion**
- We are also looking at what happens to the dynamics when a transverse field is added to the classical Ising chain, which causes all the constants of motion to instantly disappear
- **Remark:** The course has only covered spin systems, but the results extend to lattice fermionic systems.