Epilogue to Anna's course

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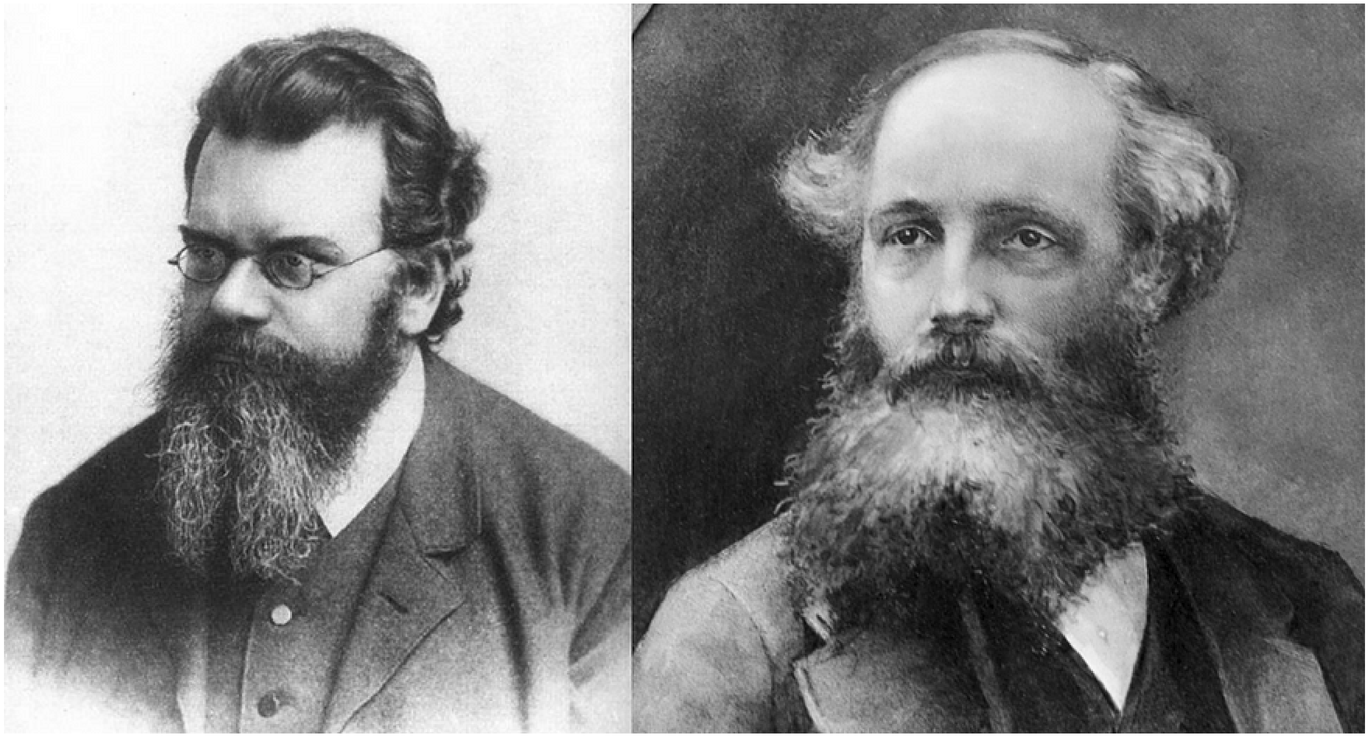
In a nutshell

Maxwell and Boltzmann



Molecular collisions lead to increase of entropy and approach to equilibrium, or in other words, collisions (and induced scattering) are the microscopic mechanism behind the thermodynamic tendency toward maximum entropy.

The results presented in the course lead to a dichotomy: Either initial setting is such that approach to equilibrium is impossible, or the system settles to equilibrium, that is, the state of maximal entropy compatible with constants of motion,

In other words, we have arrived at the structural theory of approach to equilibrium.

If microscopic scattering restores locality in the large time limit, then the approach to equilibrium is forced.

If it does not, then the setting (combination of initial state and dynamics) is non-physical from the thermodynamical perspective.

One still needs to show that in concrete physical settings the minimal restoration of locality happens! This is a very concrete problem, simple to formulate (and hopefully to study numerically). Our current understanding of dynamics of infinitely extended systems is, however, so poor that it will take a major effort to make progress in this direction.

There are basic examples to which the presented theory applies (and sheds a light on the existing results).

(1) Free Fermi gas (Lanford-Robinson, CMP, 1972. Approach to Equilibrium of Free Quantum Systems).

(2) Quantum dynamics generated by classical interactions (this includes examples of general (short and long range) Ising models discussed in Radin, JMP, 1970, Approach to Equilibrium in a Simple Model).

This is the content of the Paper IV.

Mixed field Ising chain

Discussed by Anna. Hamiltonian in the box $\Lambda = [M, N]$ is

$$H_N = \sum_{i=M}^{N-1} J_Z Z_i Z_{i+1} + \sum_{i=M}^{N} (h_X X_i + h_Z Z_i).$$

The interaction is

$$\Phi(\{i\}) = h_X X_i + h_Z Z_i, \qquad \Phi(\{i, i+1\}) = J_Z Z_i Z_{i+1},$$

otherwise $\Phi = 0$. If $h_X = 0$, then the interaction is classical, plenty of constants of motion in $\mathcal{U}_{\text{loc}}$. If $h_X \neq 0$, there is **only one** non-trivial constant of motion (up to physical equivalence),

$$E_\Phi = \frac{1}{2} J_Z (Z_0 Z_1 + Z_{-1} Z_0) + h_X X_0 + h_Z Z_0$$

Take nice initial state ω – say, equilibrium state for some finite range interaction Ψ_0. Equilibrium inverse temperature determined by

$$\omega(E_\Phi) = \omega_{\beta_{\mathrm{eq}}\Phi}(E_\Phi),$$

where $\omega_{\beta_{\mathrm{eq}}\Phi}$ is the unique equilibrium state for $\beta_{\mathrm{eq}}\Phi$. Let ω_+ be any ESS,

$$\omega_+ = \lim \frac{1}{T_n}\int_0^{T_n} \omega \circ \tau_\Phi^t \mathrm{d}t.$$

Inverse variational principle gives

$$s(\omega_+) = \inf_{\Psi \in \mathcal{B}_{\mathrm{b}}} (P(\Psi) + \omega_+(E_\Psi)).$$

If there is no potential Ψ_+ such that

$$s(\omega_+) = P(\Psi_+) + \omega_+(E_{\Psi_+}),$$

then ω_+ is non-physical.

Maximal restoration of locality: Ψ_+ exists and is finite range. Then

Theorem. $\Psi_+ = \beta_{\mathrm{eq}}\Phi$ and $\omega_+ = \omega_{\beta_{\mathrm{eq}}\Phi}$.

The point of conjectures: relaxing maximal restoration of locality, with ultimate goal of reaching minimal locality $\Psi_+ \in \mathcal{B}_{\mathrm{sd}}$. If $\Psi_+ \notin \mathcal{B}_{\mathrm{sd}}$, the system relaxed to something non-physical. Note that minimal locality requires understanding of the constant of motions in $\mathcal{B}_{\mathrm{sd}}$, so there is a max/min duality between locality and CM's.

How to establish restoration of locality for the mixed field Ising chain? work in progress.

Numerical investigation of thermalization: Banuls-Cirac-Hastings, PRL 2011, Strong and Weak Thermalization of Infinite Nonintegrable Quantum Systems

The reason for validity of the above theorem is that conjectures are established for finite range interactions in $d = 1$ and constants of motions are understood. The same theorem then holds for all non-integrable Hamiltonians of JS.

If $d > 1$, there are results on the absence of non-trivial constants of motion (Shiraishi-Tasaki 2025), and in these cases above theorem applies. Note that the conjecture R is known to hold only in the high temperature regime.

Research program

(1) Conjectures

(2) Constants of motions

(3) Study of dynamical scattering in spin systems and mechanisms that lead to restoration of locality. The JS models (say mixed field Ising model) are the starting point.

The parallel program in classical statistical mechanics has not been discussed in the course. The decoherence/ETH direction also was not discussed (and is still in a preliminary stage of development, hence Friday's working seminars).

A related conjecture

For any two finite range interaction Φ and Ψ, the limit

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{Tr} \left(e^{H_\Lambda(\Phi)} e^{H_\Lambda(\Psi)} \right)$$

exists.

This is established only in $d = 1$ (Ogata) and in the high temperature regime if $d > 1$ (Redig-Netocny). R conjecture is also proven only in those regimes. We expect that there is a relation between the conjectures on a technical level. Note the relation between this conjecture and the Quantum Stein Lemma.